MATH 134, NOTES ON ELEMENTARY TOPOLOGY TO SUPPLEMENT LECTURE 3

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1. The Spirit of This Class

The main subject of this class is "Calculus with Complex Numbers"–we will learn about complex numbers, functions whose input and output are complex numbers (so-called complex functions), and derivatives and integrals of complex functions. The advanced theoretical foundation of calculus is called "Mathematical Analysis"; for real functions it is called "Real Analysis" and for complex functions it is called "Complex Analysis." However Real Analysis (Math 130AB) is *not* a prerequisite for our class; the only prerequisite is Calculus III (Math 32). Thus we will minimize some of the technical difficulties that you would learn about in a Real Analysis class, making our class, in spirit, more a successor of Calculus III than a successor of Real Analysis.

2. Open and Closed Sets

To illustrate what I mean by "the spirit of this class," consider the book's homework problems about open and closed sets. The book asks you to:

(1) Determine if a given set is open.
(2) Determine if a given set is closed.
(3) Determine the interior points of a given set.
(4) Determine the boundary points of a given set.
(5) (etc.)

Note that the book does *not* ask you to prove your assertions (that would be "analysis"), though it still does expect you to give the correct answer. Still, proofs can be fun and would test your understanding. For example, I gave the following example in lecture, but I am only now proving my assertions:

Example. Let

$$E_3 = \{z \in \mathbb{C};\ 1 < \mathrm{Re} z < 2,\ \mathrm{Im} z = 0\}.$$

(a) E_3 is not open.

Proof. Let $a \in E_3$. Then for every $r > 0$ the open disc $D_r(a)$ is not a subset of E_3. After all, $a + i(\frac{r}{2}) \in D_r(a)$ but $a + i(\frac{r}{2}) \notin E_3$. $\square$

(b) E_3 is not closed.

Proof. The number 1 is an accumulation point (and boundary point!) of E_3 but is not in E_3. After all, for every $r > 0$ let $z = \min\{1 + \frac{r}{2},\ \frac{3}{2}\}$. Then $z \in D_r(1) \cap E_3$. □

3. The Definition of Closed Set

In class one of you pointed out that my definition of closed set seems to differ from the book's definition of closed set. We can prove, though, that the definitions are equivalent:

Lemma. A set $E \subset \mathbb{C}$ contains all of its boundary points if and only if it contains all of its accumulation points.

Proof. $\rightarrow$: Suppose E contains all of its boundary points. Let c be an accumulation point. We need to show that $c \in E$. If c is a boundary point we are done. If c is *not* a boundary point, then there exists an open disc centered at c which does not contain any points *not* in E (c being an accumulation point of E means that the disc must contain a point of E). So c is an interior point of E; in particular, $c \in E$.

$\leftarrow$: Now suppose that E contains all of its accumulation points. Let b be a boundary point. We need to show that $b \in E$. If b is an accumulation point we are done. If b is *not* an accumulation point, then there exists an open disc $D_r(b)$ centered at b such that

$$D_r(b) \cap E \subset \{b\}.$$

But since b is a boundary point of E we have $D_r(b) \cap E \neq \emptyset$, so $D_r(b) \cap E = \{b\}$. Thus $b \in E$. □

Again, this is the sort of thing you would do in an Analysis class.

4. Graphs of Complex Functions

Finally, one of you asked about how to visualize the graph of a complex function. In Math 30, we mostly (only?) looked at functions whose inputs were real numbers and whose outputs were real numbers. Thus we were able to visualize the relationship between inputs and outputs in a two-dimensional coordinate system. Complex functions are a little trickier to visualize. After all, the inputs are in the plane, and the outputs are in the plane, so putting them together would give a picture in a four-dimensional coordinate system. The standard way to picture complex functions is instead to look at inputs and outputs as "before" and "after" in the *same* plane. I will often say "z is here, and the complex function f moves it over here to $f(z)$."