# MATH 31, LECTURE 15

PROF. MICHAEL VANVALKENBURGH

We started with a quiz.

**§7.3. The Substitution Rule with (Inverse) Trig Functions.**

This is a difficult but interesting section of the book. Please keep in mind that the book and I are trying to hide pesky complications from you—so you *should be* confused by some of the fine details! For now we'll do the best we can, and you can revisit this later in life when you need it.

If you see $\sqrt{1-x^2}$ in an integral, it should remind you of the trig identity

$$\sqrt{1-\sin^2\theta} = |\cos\theta|$$

or the trig identity

$$\sqrt{1-\cos^2\theta} = |\sin\theta|.$$

So try making the substitution $x = \sin\theta$ or $x = \cos\theta$.

**Example.**

$$\int \frac{1}{\sqrt{1-x^2}}\,dx.$$

In class I simply did the substitution $x = \sin\theta$ and left it at that. But now let me give you the full story...

To be precise, the Substitution Rule with $x = \sin\theta$ is only valid on intervals where $\sin\theta$ is an *invertible* function: for example, on $(-\frac{\pi}{2}, \frac{\pi}{2})$. For those values of $\theta$, $\cos\theta$ is *positive*, so we can take away the absolute values in the identity

$$\sqrt{1-\sin^2\theta} = \cos\theta.$$

So with $x = \sin\theta$, we get $dx = \cos\theta\,d\theta$, and by the Substitution Rule:

$$\begin{aligned}\int \frac{1}{\sqrt{1-x^2}}\,dx &= \int \frac{\cos\theta}{\cos\theta}\,d\theta \\ &= \theta + C \\ &= \arcsin x + C.\end{aligned}$$

We can also choose to make the substitution $x = \cos\theta$ on the interval $(0, \pi)$ (an interval on which $\cos\theta$ is an invertible function). And on this interval $\sin\theta$ is positive, so again we can take away the absolute values and write

$$\sqrt{1 - \cos^2\theta} = \sin\theta.$$

Then we get $dx = -\sin\theta\, d\theta$, and substitution gives

$$\begin{aligned}\int \frac{1}{\sqrt{1-x^2}}\, dx &= \int \frac{-\sin\theta}{\sin\theta}\, d\theta \\ &= -\theta + D \\ &= -\arccos x + D\end{aligned}$$

(now writing $D$ as the constant of integration). This looks different than what we got last time... But not to worry! By drawing the appropriate right triangle you will see that

$$\arcsin x + \arccos x = \frac{\pi}{2},$$

so in fact we get the same thing, but with $D = \frac{\pi}{2} + C$ (it doesn't really matter, because in these problems the constant of integration represents a general additive constant).

Also, you can check your answer by differentiating, using implicit differentiation.

**The technicalities involving (1) the domains of the inverse trig functions, (2) the validity of the Substitution Rule, and (3) the resulting $\pm$ sign ambiguities are probably confusing to you. Just do your best, and I will be forgiving when I grade your tests.**

I also described what to do when you see

(1) $\sqrt{a^2 - x^2}$,
(2) $\sqrt{a^2 + x^2}$, or
(3) $\sqrt{x^2 - a^2}$

in an integral. (See the book.)

**Example.** (p.508 of the book)

$$\int_{\sqrt{2}}^{2} \frac{1}{t^3\sqrt{t^2 - 1}}\, dt.$$

We did it in class... Eventually you will need to integrate $\cos^2$, but for that use §7.2.

[Remember to change the limits of integration when you use the Substitution Rule.]