MATH 31, LECTURE 16

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At the beginning of class, I passed back the quiz and talked about it.

I also talked about "removing the absolute values" (see the Lecture 15 notes).

Example. (Similar to #14 in the book.)

$$\int \frac{1}{u\sqrt{9-u^2}}\,du.$$

In this problem, we will need to know how to evaluate

$$\int \csc\theta\,d\theta.$$

The book makes a wacky substitution. I prefer to use "partial fractions"–foreshadowing Section 7.4 of the book. Each step should make sense, even though as a whole it seems magical:

$$\begin{aligned}\int \csc\theta\,d\theta &= \int \frac{1}{\sin\theta}\,d\theta \\ &= \int \frac{\sin\theta}{\sin^2\theta}\,d\theta \\ &= \int \frac{\sin\theta}{1-\cos^2\theta}\,d\theta.\end{aligned}$$

But now we can make the substitution $x = \cos\theta$, $dx = -\sin\theta\,d\theta$ to get

$$-\int \frac{1}{1-x^2}\,dx.$$

For integrals like this, we use the algebraic identity

$$\frac{1}{1-x^2} = \frac{1}{2}\left(\frac{1}{1-x} + \frac{1}{1+x}\right).$$

(This is what is called "partial fractions.") Thus

$$-\int \frac{1}{1-x^2}\,dx = \tfrac{1}{2}\ln|1-x| - \tfrac{1}{2}\ln|1+x| + C.$$

Now substituting back, we get

$$\begin{aligned}
\int \csc\theta\, d\theta &= \tfrac{1}{2}\ln|1-\cos\theta| - \tfrac{1}{2}\ln|1+\cos\theta| + C \\
&= \tfrac{1}{2}\ln\left|\frac{1-\cos\theta}{1+\cos\theta}\right| + C \\
&\text{(we used a property of logarithms)} \\
&= \tfrac{1}{2}\ln\left|\frac{(1-\cos\theta)^2}{1-\cos^2\theta}\right| + C \\
&\text{(we multiplied the top and bottom by } (1-\cos\theta)) \\
&= \tfrac{1}{2}\ln\left|\frac{(1-\cos\theta)^2}{\sin^2\theta}\right| + C \\
&= \ln\left|\frac{1-\cos\theta}{\sin\theta}\right| + C \\
&\text{(we used another property of logarithms)} \\
&= \ln|\csc\theta - \cot\theta| + C.
\end{aligned}$$

And this is the same formula as the book.

Example. Let $a > 0$ be a fixed positive constant. Evaluate

$$\int \frac{1}{\sqrt{x^2 - a^2}}\, dx.$$

This is similar to the above example. We will need to integrate sec, but this follows the same method as above. (Compare with the book's wacky substitution in §7.2.)