MATH 31, LECTURE 2

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Summary of last time: When traveling at speed $f(t)$ mph at time t, from time a to time b we have traveled

$$L = \int_a^b f(t)\,dt \qquad \text{miles.}$$

Graphically, this number is the area under the graph of f between a and b.

In the modern world, for most functions we would just use a computer to calculate (good approximations of) integrals. But there are some special integrals that we can compute by hand, with pencil and paper.

One important method is to use the Fundamental Theorem of Calculus: Suppose that by time t we have traveled $F(t)$ miles. (That is, F calculates the distance traveled.) From Calculus I we know that the derivative at time t

$$f(t) = F'(t)$$

is the [instantaneous] speed at time t.

Question: Again, how far have we traveled between time a and time b?
Answer: We already know one way to answer: we traveled

$$\int_a^b f(t)\,dt \qquad \text{miles.}$$

But now we have another way: We traveled $F(b) - F(a)$ miles. That is, the total distance traveled up to time b, minus the total distance traveled up to time a.

Thus if f is the derivative of some function F, then

$$\begin{aligned} \int_a^b f(t)\,dt &= \int_a^b F'(t)\,dt \\ &= F(b) - F(a). \end{aligned}$$

This is half of the Fundamental of Calculus. In English words, you might say "the total change = the integral of the rate of change."

What we call "calculus" is the mathematical theory of change: the study of how much things change and the rate at which they change.

That was a conceptual review of Sections 5.1–5.4 of the book (covered in Calculus I). Please review those sections and practice doing problems. Next time we will start with concrete examples, and you need to remember the derivatives of standard elementary functions like $\cos x$, $\ln x$, x^7, etc.

Section 5.5. "The Substitution Rule."

Recall from Calculus I: the Chain Rule says

$$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x).$$

Thus by the Fundamental Theorem of Calculus (as above)

$$(*) \qquad \int F'(g(x))g'(x)\,dx = F(g(x)) + C \qquad \text{[indefinite integral]}$$

$$(**) \qquad \int_a^b F'(g(x))g'(x)\,dx = F(g(b)) - F(g(a)) \qquad \text{[definite integral].}$$

On a test, I may give you an integral and will want you to recognize it as an integral of the form $(*)$ or $(**)$, because then you can use the formula to solve the problem! The following notation is useful for recognizing integrals of this type.

Common but possibly confusing notation: Write $f = F'$ and think of $u = g(x)$ as a new variable. Then

$$\frac{du}{dx} = g'(x),$$

and we use the common but crazy-looking notation

$$du = g'(x)\,dx.$$

[Remember: $\frac{du}{dx}$ is not really a fraction but rather is a *symbol* that denotes the derivative of u with respect to x.] Then $(*)$ says

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C$$

$$||$$

$$\int f(u)\,du = F(u) + C.$$

Notice that the formula is easy to remember when written in terms of u. After all,

$$\int f(u)\,du = F(u) \qquad \textit{means} \qquad F'(u) = f(u)$$

(see p.397, §5.4, of the book).

This can be useful for calculating integrals.

On tests and quizzes, please write your answers like this:

Example. Evaluate the integral $\int x\ \cos(x^2)\,dx$.

Try substituting $u = x^2$. Then $\frac{du}{dx} = 2x$, or write $du = 2x\,dx$, or $\frac{1}{2}du = x\,dx$. Then

$$\begin{aligned} \int x\,cos(x^2)\,dx &= \frac{1}{2}\int \cos(u)\,du \\ &= \tfrac{1}{2}\sin(u) + C \\ &= \tfrac{1}{2}\sin(x^2) + C. \end{aligned}$$

We should check our answer, to make sure it is correct:

$$\frac{d}{dx}\left(\tfrac{1}{2}\sin(x^2) + C\right) = x\ \cos(x^2).$$

A few notes: Since the question is stated in terms of x, your answer should also be stated in terms of x. The choice of the variable u was an intermediate step that is your personal business. Also, when we checked our answer, we needed to use the Chain Rule. This is not surprising, since the Substitution Rule is *based on* the Chain Rule.

Example. Evaluate the integral $\int \sin^4 x\ \cos x\,dx$.

Try substituting $u = \sin x$. Then $\frac{du}{dx} = \cos x$, or write $du = \cos x\,dx$. So

$$\begin{aligned} \int \sin^4 x\ \cos x\,dx &= \int u^4\,du \\ &= \tfrac{1}{5}u^5 + C \\ &= \tfrac{1}{5}\sin^5 x + C. \end{aligned}$$

Again, we should use the Chain Rule to check our answer.

Example. Evaluate the integral $\int \frac{x+4}{x^2+1}\,dx$.

We split the integral into the sum of two integrals and treat each one separately:

$$\int \frac{x+4}{x^2+1}\,dx = \int \frac{x}{x^2+1}\,dx + 4\int \frac{1}{x^2+1}\,dx$$

$$\left(\text{try } u = x^2+1, \text{ so } \tfrac{1}{2}\,du = x\,dx\right)$$

$$\begin{aligned} &= \frac{1}{2}\int \frac{1}{u}\,du + 4\int \frac{1}{x^2+1}\,dx \\ &= \tfrac{1}{2}\ln(u) + 4\arctan x + C \\ &= \tfrac{1}{2}\ln(x^2+1) + 4\arctan x + C. \end{aligned}$$

Note: There is often more than one substitution that will help evaluate the integral. In the above example, $u = x^2$ would also lead to the correct answer. What we want is any substitution that turns the integral into an equivalent integral that is easier to evaluate.