MATH 31, LECTURE 21

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Today we covered §6.5 (Average Value of a Function), and I ended by doing a review of $f(x) = e^x$ and $g(y) = \ln y$. I started from the beginning, talking about the basic properties of exponential functions.

Exponentials. Let $a > 0$. Then by definition we have $a^3 = a \cdot a \cdot a$. Also,

$$a^{15} = a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a.$$

It's obviously better to write it as a^{15}.

For any positive integer n, we have by definition

$$a^n = a \cdot a \cdots a \qquad (n \text{ times}).$$

From just this much, we can figure out the basic properties of exponentials:

$$a^{m+n} = a^m a^n,$$

$$(a^m)^n = a^{mn}.$$

In fact, the second property can be understood by writing

$$\begin{matrix} a & a & \cdots & a \\ a & a & \cdots & a \\ \vdots & & & \vdots \\ a & a & \cdots & a \end{matrix}$$

where the array has m columns and n rows. For example,

$$(a^4)^2 = (a \cdot a \cdot a \cdot a)^2 = \begin{matrix} (a & \cdot a & \cdot a & \cdot a) \\ \cdot(a & \cdot a & \cdot a & \cdot a) \end{matrix} = a^8.$$

We also use the notation, again for n a positive integer,

$$a^{-n} = \frac{1}{a^n}.$$

We then have the property

$$a^{m-n} = \frac{a^m}{a^n}.$$

Leap of faith: It is a fact that we can also make sense of these properties for any numbers x and y:

$$a^x a^y = a^{x+y}$$
$$a^{x-y} = \frac{a^x}{a^y}$$
$$(a^x)^y = a^{xy}.$$

[I drew what the graph of $f(x) = a^x$ looks like when $a > 1$ and when $0 < a < 1$.]

The number $e = 2.718281828459045\ldots$ is the number such that $f(x) = e^x$ has the property

$$f'(x) = f(x).$$

That is, the instantaneous slope of the graph at x is equal to the function's value at x. For example, the slope at 0 is 1.

The Inverse of $f(x) = e^x$. The function $f(x) = e^x$, as a function from $(-\infty, \infty)$ to $(0, \infty)$, is an *invertible* function. Its inverse is called the *natural logarithm* and is written $g(y) = \ln y$. Being the inverse function means:

$$g(f(x)) = x \quad \text{for all } x\text{, and}$$
$$f(g(y)) = y \quad \text{for all } y > 0.$$

Using the fact that it is an inverse, we can get the basic properties of the logarithm from the basic properties of the function $f(x) = e^x$. For example:

We write $f(x_1) = y_1$, so that $x_1 = g(y_1)$, and $f(x_2) = y_2$, so that $x_2 = g(y_2)$. Then

$$\begin{aligned} g(y_1 y_2) &= g(f(x_1)f(x_2)) \\ &= g(f(x_1 + x_2)) \\ &= x_1 + x_2 \\ &= g(y_1) + g(y_2). \end{aligned}$$

That is, using the fact that f and g are inverses and that $f(x_1 + x_2) = f(x_1)f(x_2)$, we have *shown* that

$$g(y_1 y_2) = g(y_1) + g(y_2).$$

That is,

$$\ln(y_1 y_2) = \ln y_1 + \ln y_2.$$

This is one of the basic properties of logarithms that I want you to know, and now you know *why* it is true. In colorful language, "$f(x) = e^x$ turns addition into multiplication, and the inverse $g(y) = \ln y$ turns multiplication into addition."

Similarly, you can show that

$$\ln(y^n) = n \ln y.$$

To summarize, the basic properties of the natural logarithm are:

$$e^{\ln y} = y \qquad \text{for all } y > 0$$

$$\ln(e^x) = x \qquad \text{for all } x$$

(the above two properties exactly *say* that they are inverse functions of each other!)

$$\ln(ab) = \ln a + \ln b$$

$$\ln(\tfrac{a}{b}) = \ln a - \ln b$$

$$\ln(y^n) = n \ln y$$

and we can use implicit differentiation to show that

$$\frac{d}{dy} \ln y = \frac{1}{y}.$$

One reason why exponentials and logarithms are useful: The function $f(x) = e^x$ is difficult to graph for large $|x|$. As $x \to -\infty$ it gets *very* close to 0, and as $x \to \infty$ it grows *very* fast. For example,

$$e^{-10} \approx 0.00004534$$

and

$$e^{10} \approx 22,000.$$

If we go out even a little further, it's much, much bigger:

$$e^{15} \approx 3,269,017.$$

This is why in class I only graph $f(x) = e^x$ for small values of x.

A better way to visualize functions that grow this fast is to graph them on a *logarithmic scale*. One famous example of a logarithmic scale is the *Richter scale* for quantifying the energy released by an earthquake. Here are some examples from wikipedia:

Richter scale	Joule equivalent	Example
0.2	130×10^3 J	hand grenade
1.4	9.8×10^6 J	small construction blast
3.0	2.0×10^9 J	Oklahoma City bombing
4.0	63×10^9 J	Johannesburg earthquake, 2013
5.0	2.0×10^{12} J	UK earthquake, 2008
6.0	63×10^{12} J	Hiroshima atomic bomb
6.7	710×10^{12} J	Northridge earthquake, 1994
6.9	1.4×10^{15} J	SF earthquake, 1989
8.0	63×10^{15} J	SF earthquake, 1906
8.6	500×10^{15} J	Sumatra earthquake, 2012
13.0	420×10^{21} J	Yucatán meteor impact, 65 million years ago

"Only a 13.0 on the Richter scale????"

Note: If it were on a linear scale, a 13.0 would be 65 times bigger than a 0.2. But the Richter scale is *not* a linear scale. The meteor impact that likely killed off the land dinosaurs was much, much bigger than 65 hand grenades.