

MATH 31, LECTURE 27

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Last time: For $f(x) = x^2$ on $[0, 2]$, with $n = 4$ subintervals of equal length, we have the following errors for the four different methods:

$$\begin{aligned}E_{\text{left}} &= \frac{11}{12} \\E_{\text{right}} &= -\frac{13}{12} \\E_{\text{midpt}} &= \frac{1}{24} \\E_{\text{trap}} &= -\frac{1}{12}.\end{aligned}$$

So for this example the Midpoint Rule gives the best approximation.

The above example is a bit silly because we can easily find the *exact* value of the integral, using the Fundamental Theorem of Calculus. The only reason we did approximation methods for this example was to illustrate the techniques, to compute everything by hand. In real-world problems (likely using a computer to implement the Midpoint Rule or Trapezoidal Rule), we probably don't know the *exact* error because we probably don't know the exact value of the integral! It is, however, always possible to *estimate* the error, given a bound on the second derivative:

Error Bounds. Suppose $|f''(x)| \leq K$ for $a \leq x \leq b$. Then, dividing $[a, b]$ into n equal subintervals,

$$|E_{\text{trap}}| \leq \frac{K(b-a)^3}{12n^2}, \quad \text{and}$$

$$|E_{\text{midpt}}| \leq \frac{K(b-a)^3}{24n^2}.$$

These error bounds are valid for *any* (twice-differentiable) function f . If you take Math 130AB (Real Analysis) you will learn the skills needed to prove these error bounds.

Again, these error bounds are valid for *any* (twice-differentiable) function f . But for fun let's see what they say for a special case: our simple example $f(x) = x^2$ on $[0, 2]$ with $n = 4$ subintervals. For this example, $f''(x) = 2$, so the best (smallest) K is $K = 2$. Then the error bounds say:

$$|E_{\text{trap}}| \leq \frac{1}{12}, \quad \text{and}$$

$$|E_{\text{midpt}}| \leq \frac{1}{24}.$$

Ok, that agrees with what we got before!

Now let's look at a more complicated example:

Application. Suppose we want to approximate the integral

$$\int_0^3 e^{-x^2} dx$$

using the Midpoint Rule, with an error less than 10^{-6} . That is, we want an n so that

$$|E_{\text{midpt}}| \leq 10^{-6}.$$

Using the error bound, with $a = 0$, $b = 3$, and $f(x) = e^{-x^2}$, we have

$$\begin{aligned} f(x) &= e^{-x^2} \\ f'(x) &= -2xe^{-x^2} \\ f''(x) &= 2(2x^2 - 1)e^{-x^2}. \end{aligned}$$

To find the *best* (smallest) K so that $|f''(x)| \leq K$ for all $0 \leq x \leq 3$, we can use optimization techniques from Calc. I (remember “find the absolute maximum,” etc.?). But let's be lazy and make the computer do the work. We use the crude estimate

$$|f''(x)| \leq 2 \cdot 17 \cdot 1 = 34 \quad \text{for all } 0 \leq x \leq 3,$$

which shows that we can take $K = 34$. (We can likely find a better (smaller) K , but this one is ok.) we then choose n so that

$$\frac{K(b-a)^3}{24n^2} = \frac{34 \cdot 27}{24n^2} = \frac{17 \cdot 9}{4n^2} \leq 10^{-6}.$$

That is, we take

$$\sqrt{\frac{17 \cdot 9 \cdot 10^6}{4}} = 1500\sqrt{17} \leq n.$$

So we can take, for example, $n = 6185$ to guarantee that the error is less than 10^{-6} .

(See what I mean by making the computer do the work? Now the computer needs to split the interval into 6185 equal subintervals and then needs to use the Midpoint Rule.)

There is a fifth approximation method, “Simpson’s Rule,” which uses parabolas to approximate the graph of f . And we have the following error bound:

Error Bound for Simpson’s Rule: If $|f^{(4)}(x)| \leq K$ for all $a \leq x \leq b$, then

$$|E_{\text{Simp}}| \leq \frac{K(b-a)^5}{180n^4}.$$

For large n this is the best method we’ve seen. Also, for fun, note that for $f(x) = x^2$ we can take $K = 0$, which says that Simpson’s Rule gives the exact answer for that function. (This makes sense from what I said above: it is possible to approximate a parabola by a parabola and have no error!)

§7.8. Improper Integrals.

We ended class with a preliminary discussion of improper integrals.