MATH 31, LECTURE 28

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§7.8. Improper Integrals.
In this lecture I mostly stuck to the basics, as covered in the textbook. But here is something extra:

Example. Is the improper integral
$$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$
convergent or divergent?

The function $f(x) = \frac{1}{\sqrt{x}}$ has a discontinuity at $x = 0$. Let $0 < \epsilon < 1$. Then
$$\begin{aligned}\int_\epsilon^1 \frac{1}{\sqrt{x}}\,dx &= [2\sqrt{x}]_\epsilon^1 \\ &= 2 - 2\sqrt{\epsilon} \\ &\to 2 \qquad \text{as } \epsilon \to 0^+.\end{aligned}$$
So the improper integral converges:
$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\epsilon\to 0^+} \int_\epsilon^1 \frac{1}{\sqrt{x}}\,dx = 2.$$

That is a fairly standard example. But I want to point out, for fun, that the "area under the curve"
$$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$
is the same as the area under the curve
$$\begin{aligned}\int_0^1 1\,dt + \int_1^\infty t^{-2}\,dt &= 1 + \lim_{L\to\infty} \int_1^L t^{-2}\,dt \\ &= 1 + \lim_{L\to\infty}\left[-\frac{1}{t}\right]_1^L \\ &= 2.\end{aligned}$$

The relationship between x and t is $t = \frac{1}{\sqrt{x}}$, so that

$$(\frac{1}{\sqrt{x}}, x) = (t, \frac{1}{t^2}).$$

(I think it will make sense if you draw the graphs.)