

MATH 31, NOTES ON SEQUENCES AND SERIES

PROF. MICHAEL VANVALKENBURGH

The purpose of these notes: to *supplement* the book and lecture notes. See the book and lecture notes for examples.

1. SEQUENCES

A **sequence** is simply an ordered list of numbers:

$$a_1, \quad a_2, \quad a_3, \quad a_4, \quad \dots$$

We also write a sequence of numbers as:

$$\{a_n\}_{n=1}^{\infty}.$$

The most important concept in calculus is the concept of a *limit*. After all, a derivative is defined as a certain limit and an integral is defined as a certain limit. But here we will talk about limits of *sequences*, which should actually be easier to understand than limits of *functions*.

Casual/imprecise definition: L is the limit of a sequence $\{a_n\}_{n=1}^{\infty}$ if the sequence “gets closer and closer to L .” This is not the most accurate way to define it. After all, doesn’t $\{1 - \frac{1}{n}\}_{n=1}^{\infty}$ get “closer and closer to 2”? We should be more precise.

Let (c, d) be some open interval. We say that a sequence $\{a_n\}_{n=1}^{\infty}$ is *eventually* in (c, d) if there exists some N such that a_n is in (c, d) for all $n \geq N$. (The terms $a_1, a_2, \dots, a_{N-1}$ may or may not be in (c, d) , but all the rest *are* in (c, d) .)

[Aside: There is also a mathematical definition of *frequently*. Can you guess what it is?]

We say that the sequence $\{a_n\}_{n=1}^{\infty}$ *converges* to L (equivalently, that L is the limit of the sequence), if:

FOR ALL $\epsilon > 0$, THE SEQUENCE $\{a_n\}_{n=1}^{\infty}$ IS EVENTUALLY IN $(L - \epsilon, L + \epsilon)$.

That is, no matter how small of an interval we put around L , the sequence is eventually in that interval.

This is a good, precise, mathematical definition. You should be able to use this definition to show that

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

There are a number of method for proving that a sequence converges:

- (1) Use the Squeeze Theorem. See the book for details. But as one simple example, if you can show that

$$0 \leq |a_n - L| \leq \frac{1}{n},$$

then you have shown that $\lim_{n \rightarrow \infty} a_n = L$. Can you see what is being squeezed?

- (2) Use the Monotone Sequence Theorem. If a sequence is increasing and bounded above, then it *must* converge to something. Similarly, if a sequence is decreasing and bounded below, then it *must* converge to something.
- (3) If $f(x)$ is a function such that $\lim_{x \rightarrow \infty} f(x) = L$, then as a special case we must have

$$\lim_{n \rightarrow \infty} f(n) = L.$$

Once you turn it into a problem about *functions*, you might be able to use L'Hôpital's Rule.

- (4) There might be more ways, but those are the most basic.

Example. Find

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n^2 + 1}}{4n^3 + 2}.$$

The idea is to look at the *dominant* terms. The dominant term in the numerator is $\sqrt{n^2} = n$, and the dominant term in the denominator is $4n^3$. After all, the philosophical question of our time is: would Bill Gates bend over to pick up \$2? The \$2 is negligible compared to $4n^3$ when n is large.

$$\lim_{n \rightarrow \infty} \frac{n}{4n^3} = \lim_{n \rightarrow \infty} \frac{1}{4n^2} = 0.$$

To be more careful, we can do some algebra, dividing both the numerator and denominator by n :

$$\begin{aligned}\frac{\sqrt{n^2 + 1}}{4n^3 + 2} &= \frac{\frac{1}{n}\sqrt{n^2 + 1}}{\frac{1}{n}(4n^3 + 2)} \\ &= \frac{\sqrt{1 + \frac{1}{n^2}}}{4n^2 + \frac{2}{n}}.\end{aligned}$$

Now to be completely careful, we can use the Squeeze Theorem:

$$0 \leq \frac{\sqrt{1 + \frac{1}{n^2}}}{4n^2 + \frac{2}{n}} \leq \frac{2}{4n^2} \rightarrow 0.$$

Note: We are not talking about series yet. If you get a question asking “Does this sequence converge?” you probably will NOT want to integrate.

2. SERIES

An *infinite series* is a formal summation:

$$a_1 + a_2 + a_3 + a_4 + \cdots$$

The modern way of writing this is to use *summation notation* (also called “sigma notation”):

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots.$$

Note that we can “re-index.” For example, we can also write

$$\sum_{n=5}^{\infty} a_{n-4} = a_1 + a_2 + a_3 + a_4 + \cdots.$$

Also,

$$\sum_{n=0}^{\infty} a_n = a_0 + \sum_{n=1}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + \cdots.$$

The most important concept for infinite series: to look at the *sequence of partial sums*. Given an infinite series $\sum_{n=1}^{\infty} a_n$, the first partial sum is $s_1 = a_1$. The second partial sum is $s_2 = a_1 + a_2$, etc. We say that a series *converges to* L if the sequence of partial sums $\{s_n\}_{n=1}^{\infty}$ converges to L . Then we can make sense of “the sum of the series is L .”

To be continued...