Math 32: Double Integrals in Polar Coordinates

Prof. Michael VanValkenburgh

To make it easier to type and easier to read, this handout will focus on the computational aspects of integration in polar coordinates. In class we will discuss the more conceptual parts of the theory, with pictures.

1 Background on Polar Coordinates

So far we have described points in the plane by their *rectangular* coordinates (also known as "Cartesian coordinates" in honor of René Descartes). In rectangular coordinates, a point P is described by the ordered pair (x, y), where x is the horizontal component and y is the vertical component (measured from the origin).

It is often more convenient to describe a point in the plane by its *polar* coordinates (r, θ). This is an ordered pair, where now the first coordinate is the distance from P to the origin, and θ is the angle P makes with the positive x-axis. Note that (r, θ) and $(r, \theta + 2\pi)$ represent the same point, and $(0, \theta)$ represents the origin for all real numbers θ.

Example. The disc of radius 1 centered at the origin is described in rectangular coordinates as:

$$D = \{(x, y) : \ -1 \leq x \leq 1,\ -\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}\}.$$

In polar coordinates, it has a simpler description:

$$D = \{(r, \theta) : \ 0 \leq r \leq 1,\ 0 \leq \theta < 2\pi\}.$$

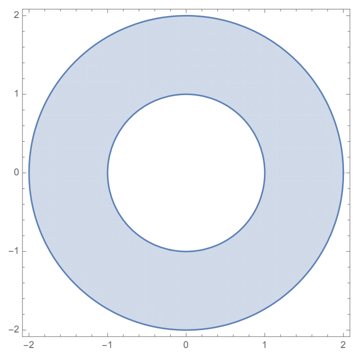
Here is how rectangular coordinates (x, y) are expressed in terms of polar coordinates (r, θ):

$$x = r \cos \theta, \qquad y = r \sin \theta.$$

And here is how polar coordinates (r, θ) are expressed in terms of rectangular coordinates (x, y):

$$\theta = \arctan(y/x), \qquad r = \sqrt{x^2 + y^2}.$$

Example. How do we describe the following annulus?



It is easiest to describe it using polar coordinates:

$$\{(r, \theta) : \ 1 \le r \le 2, \ 0 \le \theta < 2\pi\}.$$

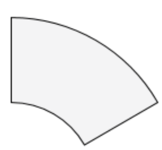
Try to describe it in rectangular coordinates. It's not so easy!

2 Double Integrals in Polar Coordinates

A "polar rectangle" is a set expressed in polar coordinates as

$$R = \{(r, \theta) : \ a \le r \le b, \ \alpha \le \theta \le \beta\}$$

for some fixed a, b, α, β. It looks something like this:



Here is the rule for integrating over R (I will give details in class): The integral of a function $f(x,y)$ over R is

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

See how we are using $x = r\cos\theta$ and $y = r\sin\theta$.
Note: **There is an "extra" factor r.** In class I will explain where it comes from.

Example. Let consider the disc $D = \{(x,y) : \ x^2 + y^2 \le 1\}$ and the function $f(x,y) = x^2 + y^2$. In polar coordinates we can write D as:

$$D = \{(r,\theta) : \ 0 \le \theta \le 2\pi, \ 0 \le r \le 1\},$$

and we can write f as:

$$f(x,y) = f(r\cos\theta, r\sin\theta) = (r\cos\theta)^2 + (r\sin\theta)^2 = r^2.$$

Thus we have

$$\begin{aligned}\iint_D f(x,y)\,dA &= \int_0^{2\pi} \int_0^1 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta \\ &= \int_0^{2\pi} \int_0^1 r^2\,r\,dr\,d\theta \\ &= 2\pi \int_0^1 r^3\,dr \\ &= \frac{\pi}{2}.\end{aligned}$$

Try doing the integral in rectangular coordinates. It is possible, but not as easy!

We just saw how to integrate over a polar rectangle. It is also possible to integrate over more general regions. For example, if R is the plane region described in polar coordinates by:

$$\alpha \le \theta \le \beta \qquad \text{and} \qquad g_1(\theta) \le r \le g_2(\theta),$$

where $g_1(\theta)$ and $g_2(\theta)$ are two continuous functions of θ, then we use the formula

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

Don't forget the "extra" factor of r!

Example. Evaluate the double integral

$$\int_0^1 \int_{x^2}^{x} (x^2 + y^2)^{-1/2}\, dy\, dx$$

by first changing to polar coordinates.

The first step: Sketch the region of integration and express it in terms of polar coordinates:

$$0 \leq \theta \leq \frac{\pi}{4} \qquad 0 \leq r \leq \,?$$

Try it for yourself!