

INTRODUCTION TO LINE INTEGRALS

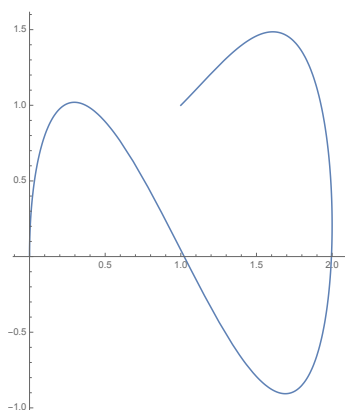
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Last week we discussed triple integrals. This week we will study a new topic of great importance in mathematics and physics: “line integrals.”

1. LINE INTEGRALS WITH RESPECT TO ARCLENGTH

1.1. **In the Plane.** Let C be a curve in $\mathbb{R}^2$, parametrized by

$$\vec{r}(t) = \langle x(t), y(t) \rangle, \quad a \leq t \leq b.$$

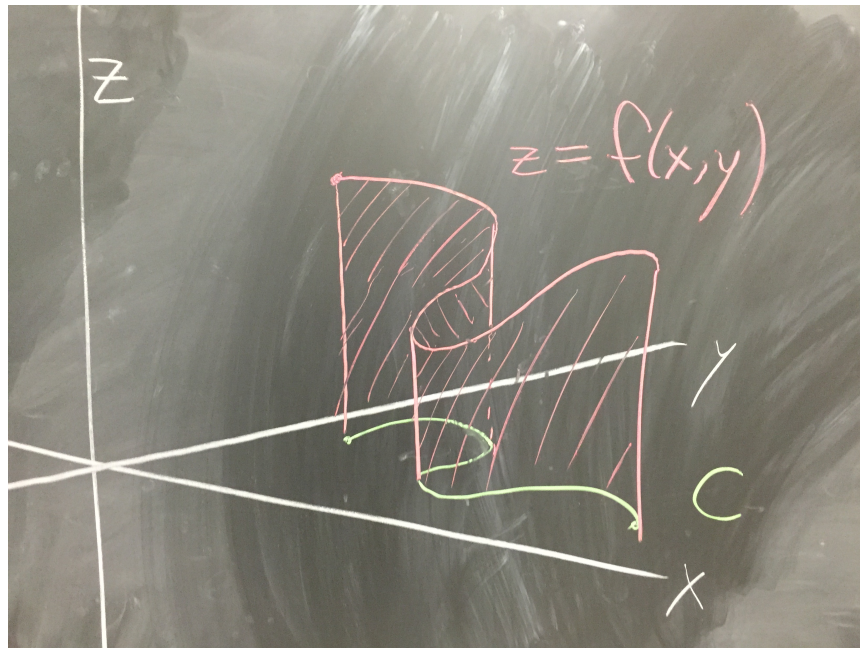


The curve pictured is given by $\vec{r}(t) = \langle \sin(t) + 1, e^{-t} + \sin(2t) \rangle$, $0 \leq t \leq 3\pi/2$.

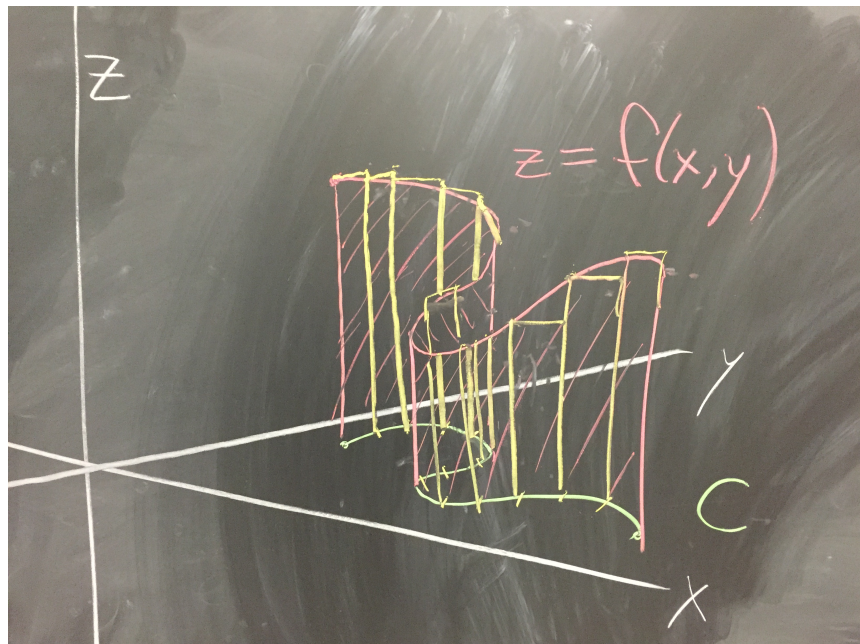
Remember that the curve does *not* need to satisfy the vertical line test, because while $\vec{r}: [a, b] \rightarrow \mathbb{R}^2$ is a function of t , the y value does *not* need to be a function of x .

Now let $f(x, y)$ be a function defined on C .

Question: What is the area of the “red ribbon,” above the curve C and below the graph of f ?



We set up the area as an integral, in the usual way. Split the “time” interval $[a, b]$ into n equal pieces. Then the curve C is split into n (possibly unequal) pieces. We then have n “wavy rectangles,” with bases along the curve C .



The height of a “wavy rectangle” is

$$\text{height} = f(\vec{r}(t_j))$$

and the length of the base is

$$\text{length} \approx \|\vec{r}'(t_j)\| \Delta t.$$

After all, from the physics perspective, it is:

$$\|\vec{r}'(t_j)\| \Delta t = \text{speed} \times \text{time} = \text{distance traveled}.$$

Adding up the areas of the rectangles, and taking the limit as $n \rightarrow \infty$ as usual, we find that the area of the red ribbon is

$$\begin{aligned} (*) \quad \int_C f(x, y) \, ds &\stackrel{\text{def.}}{=} \int_a^b f(\vec{r}(t)) \|\vec{r}'(t)\| \, dt \\ &= \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt. \end{aligned}$$

This is called “a line integral with respect to arclength.” After all, in the special case when $f \equiv 1$, the integral is

$$\int_C ds = \text{length of the curve } C.$$

You can think of it as the area of the ribbon with constant height 1 whose base is along the curve C . So the area is

$$1 \times (\text{length of } C) = \text{length of } C.$$

The definition $(*)$ resembles the Substitution Rule from Calc II:

$$\int_a^b g(u(x)) \frac{du}{dx} \, dx = \int_{u(a)}^{u(b)} g(u) \, du.$$

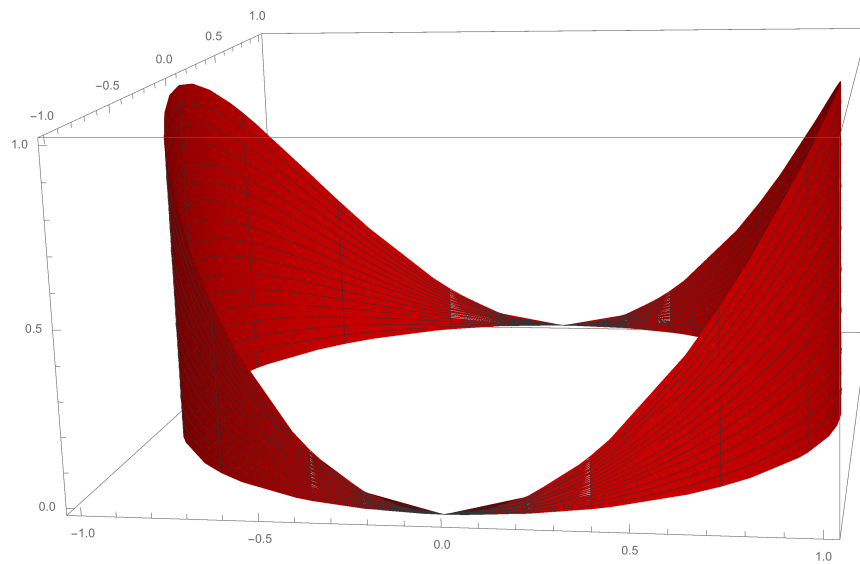
Important Fact. It is not too hard to prove that the integral $\int_C f(x, y) \, ds$ does not depend on the *way* we parametrized the curve C . That is, the geometric *area* of the red ribbon above does not depend on how “fast” the parametrization $\vec{r}(t)$ travels along the curve. We illustrate this with the following example:

Example. Let C be the curve $x^2 + y^2 = 1$, and let $f(x, y) = x^2$. [The graph of f is the parabolic sheet $z = x^2$.] Then $\int_C f(x, y) \, ds$ is the area of following red ribbon:

What I drew in class:



Using Mathematica:



To calculate the area of the ribbon, we parametrize the circle in the usual way:

$$\vec{r}(t) = \langle \cos t, \sin t \rangle, \quad 0 \leq t \leq 2\pi.$$

Then we have

$$\vec{r}'(t) = \langle -\sin t, \cos t \rangle, \quad \|\vec{r}'(t)\| = 1.$$

So the area is

$$\begin{aligned} \int_C f(x, y) \, ds &= \int_0^{2\pi} f(\vec{r}(t)) \|\vec{r}'(t)\| \, dt \\ &= \int_0^{2\pi} f(\cos t, \sin t) \cdot 1 \, dt \\ &= \int_0^{2\pi} \cos^2 t \, dt \\ &= \pi. \end{aligned}$$

We could have also parametrized the circle by

$$\vec{r}(t) = \langle \cos(2t), \sin(2t) \rangle, \quad 0 \leq t \leq \pi,$$

(tracing out the same curve, but twice as fast). Then we have

$$\vec{r}'(t) = \langle -2\sin(2t), 2\cos(2t) \rangle, \quad \|\vec{r}'(t)\| = 2.$$

So the area is

$$\begin{aligned} \int_C f(x, y) \, ds &= \int_0^\pi f(\vec{r}(t)) \|\vec{r}'(t)\| \, dt \\ &= \int_0^\pi f(\cos(2t), \sin(2t)) \cdot 2 \, dt \\ &= \int_0^\pi \cos^2(2t) \cdot 2 \, dt \\ &= \int_0^{2\pi} \cos^2 u \, du \\ &= \pi, \end{aligned}$$

where we did the substitution $u = 2t$. Thus, as we expected, the answer does *not* depend on the particular way that we parametrized the curve.

Moral. We need the factor $\|\vec{r}'(t)\|$ in order to get the same answer no matter how we choose to parametrize the curve. “If you travel faster, everything looks skinnier, and we need to account for that.”

As a cultural aside, In Ray Bradbury's science fiction novel *Fahrenheit 451*, billboards are 200 feet long:

Ray Bradbury

"I rarely watch the 'parlor walls' or go to races or Fun Parks. So I've lots of time for crazy thoughts, I guess. Have you seen the two-hundred-foot-long billboards in the country beyond town? Did you know that once billboards were only twenty feet long? But cars started rushing by so quickly they had to stretch the advertising out so it would last."

"I didn't know that!" Montag laughed abruptly.

"Bet I know something else you don't. There's dew on the grass in the morning."

He suddenly couldn't remember if he had known this or not, and it made him quite irritable.

"And if you look"—she nodded at the sky—"there's a man in the moon."

He hadn't looked for a long time.

They walked the rest of the way in silence, bare thoughtful

To a passenger in a very-fast-moving car, the billboard would look shorter than it really is. So again, if you're measuring sizes of things from inside a fast-moving car, you need to account for your speed.

Example. Find the length L of the curve given by

$$\vec{r}(t) = \langle e^t + e^{-t}, 5 - 2t \rangle, \quad 0 \leq t \leq 3.$$

We have

$$\vec{r}'(t) = \langle e^t - e^{-t}, -2 \rangle, \quad \text{and} \quad \|\vec{r}'(t)\| = e^t + e^{-t}.$$

Thus

$$\begin{aligned} L &= \int_C ds = \int_0^3 \|\vec{r}'(t)\| dt \\ &= \int_0^3 (e^t + e^{-t}) dt \\ &= e^3 - e^{-3}. \end{aligned}$$

1.2. In Three-Dimensional Space. It works the same way for curves in $\mathbb{R}^3$. Let C be a curve in $\mathbb{R}^3$ parametrized by

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle, \quad a \leq t \leq b.$$

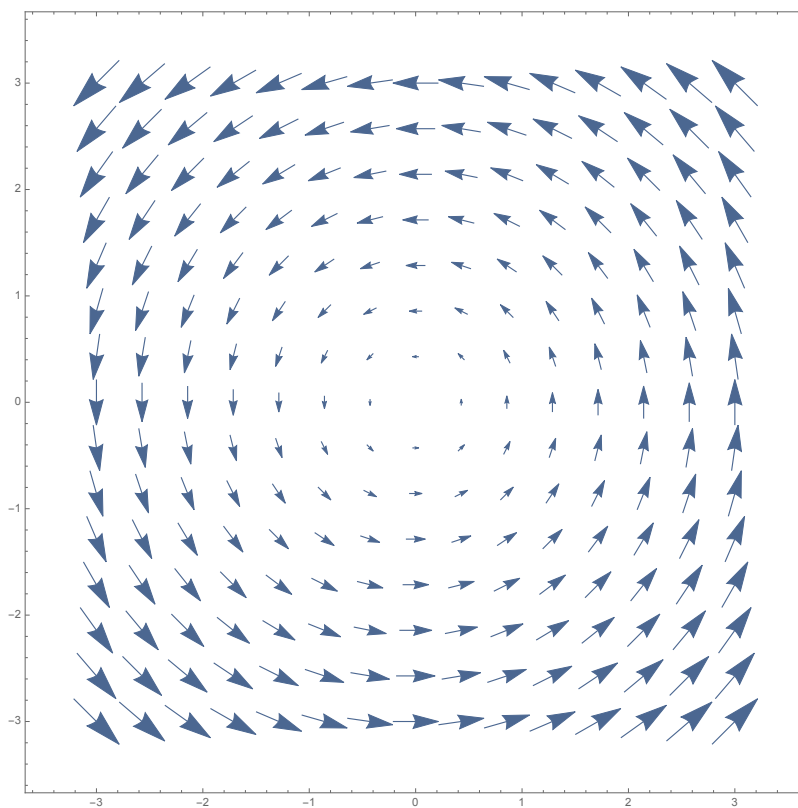
Then

$$\int_C f(x, y, z) ds \stackrel{\text{def.}}{=} \int_a^b f(\vec{r}(t)) \|\vec{r}'(t)\| dt.$$

2. LINE INTEGRALS OF VECTOR FIELDS

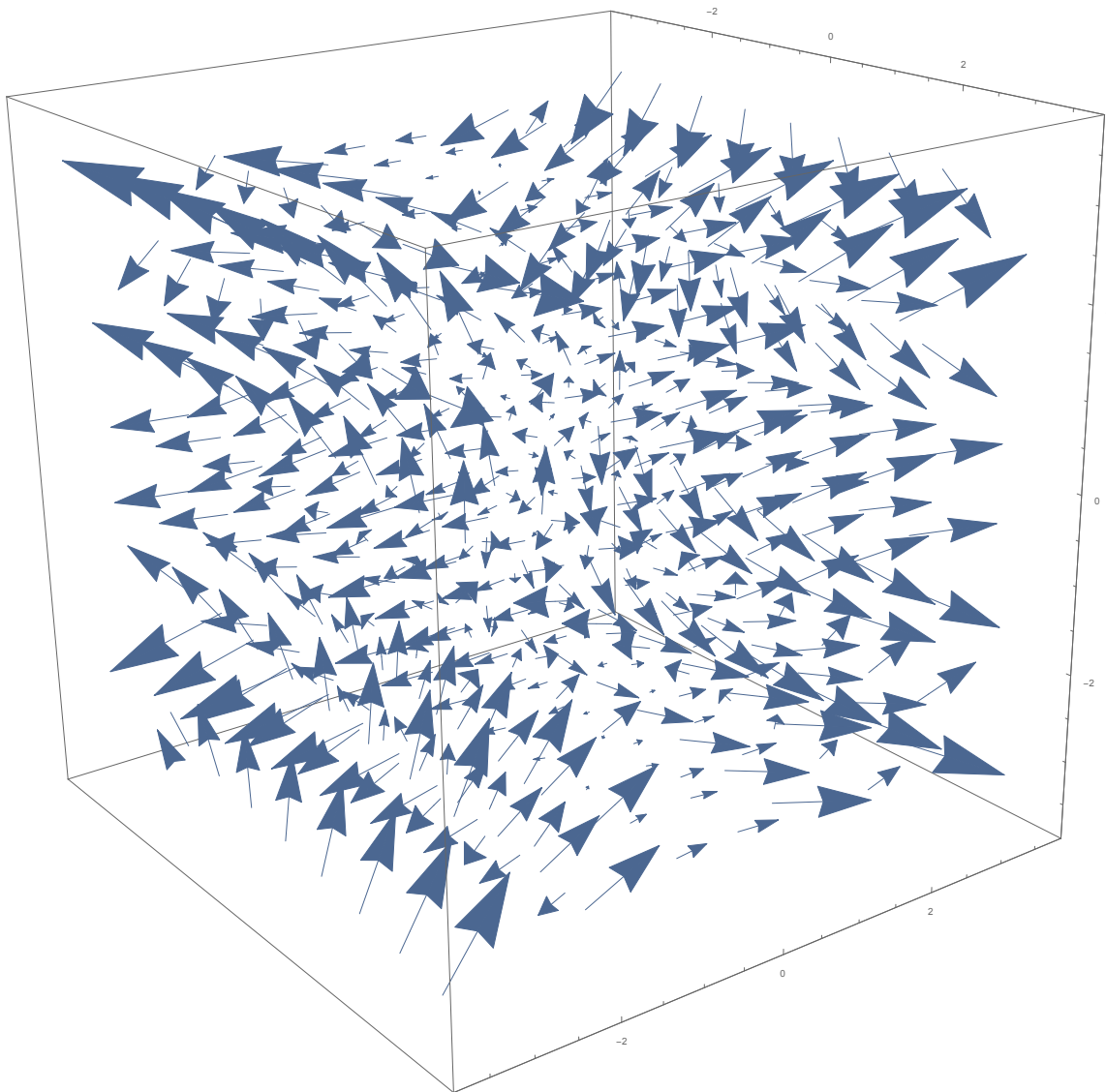
2.1. Vector Fields. A *vector field* is a function $\vec{F}$ that assigns a vector to every point in its domain. To visualize a vector field, at every point P we draw the assigned vector (arrow) with its base at P .

Example. $\vec{F}(x, y) = \langle -y, x \rangle$ is a vector field on $\mathbb{R}^2$.



You can think of it as the velocity vector field of a whirlpool, or a tornado...

Example. $\vec{F}(x, y, z) = \langle y, z \cos(yz) + x, y \cos(yz) \rangle$ is a vector field on $\mathbb{R}^3$. I didn't draw it in class, but here it is, using Mathematica:



It's easy to plot using Mathematica! Here's the code for this example:

```
In[8]:= VectorPlot3D[{y, z * Cos[y * z] + x, y * Cos[y * z]}, {x, -1, 1}, {y, -1, 1}, {z, -1, 1}]
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Example. If f is a real-valued function, then its gradient $\vec{F} = \nabla f$ is a vector field. For example, if

$$f(x, y, z) = xy + \sin(yz),$$

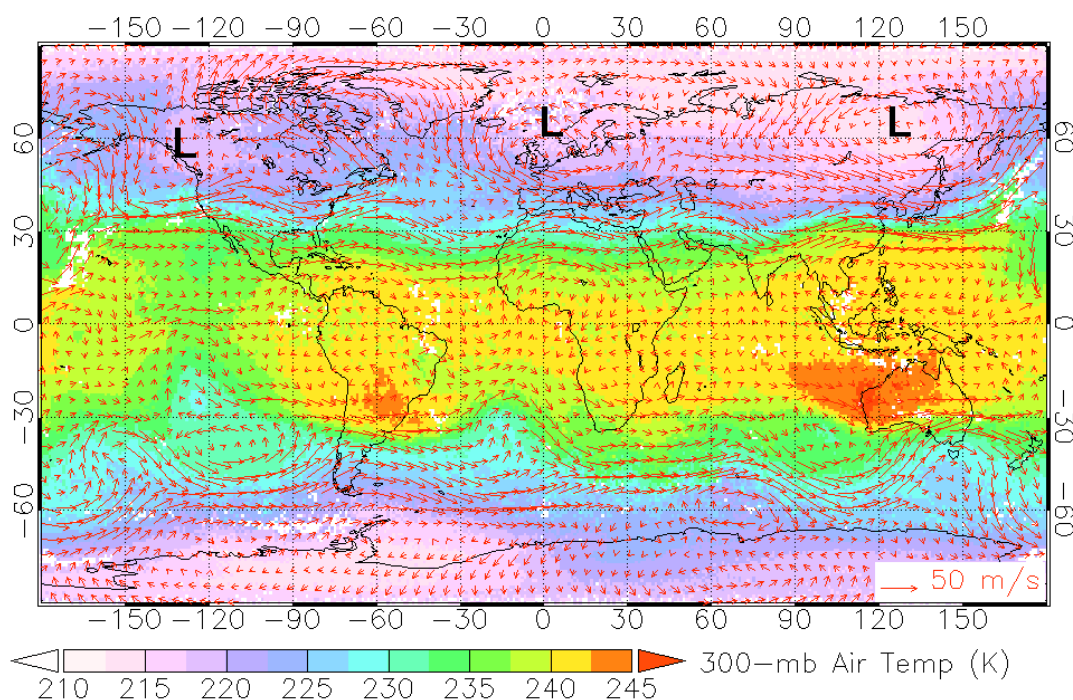
then

$$\nabla f(x, y, z) = \langle y, z \cos(yz) + x, y \cos(yz) \rangle$$

is a *gradient vector field*.

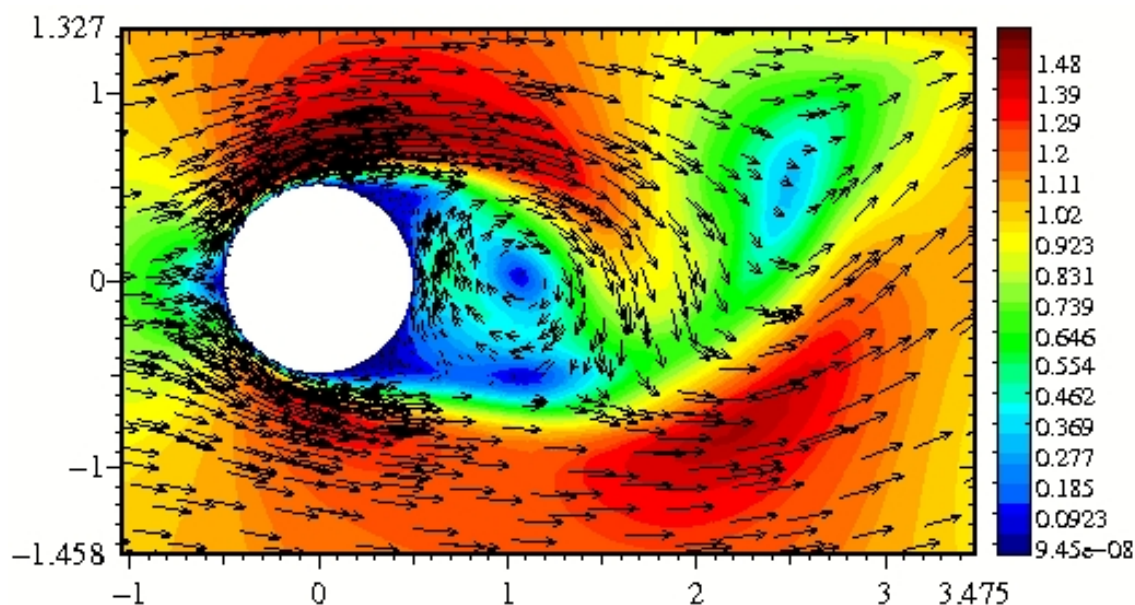
Example. Here is a wind vector field, courtesy of NASA:

<http://disc.sci.gsfc.nasa.gov/featured-items/airs-monitors-cold-weather>



Example. And here is a vector field coming from fluid flow:

<http://math.unice.fr/~rpas/>



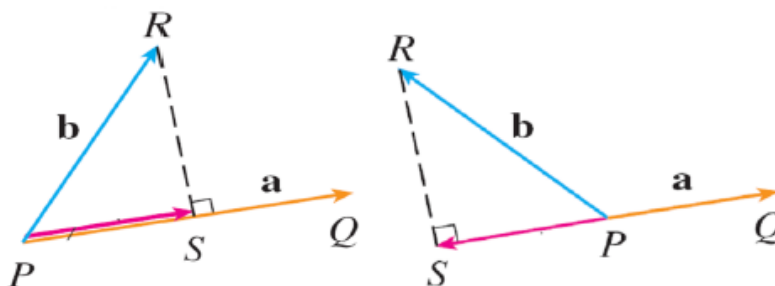
Note that it contains a little copy of the “tornado” vector field from earlier (but going clockwise this time).

2.2. Line Integrals of Vector Fields. We start with a basic fact from §2.3 of OpenStax (§12.3 of Stewart):

Basic Fact. The component of $\vec{b}$ in the direction of $\vec{a}$ is

$$\text{comp}_{\vec{a}} \vec{b} = \frac{\vec{b} \cdot \vec{a}}{\|\vec{a}\|} = \|\vec{b}\| \cos \theta,$$

where θ is the angle between $\vec{a}$ and $\vec{b}$. The quantity $\text{comp}_{\vec{a}} \vec{b}$ is a scalar which may be positive or negative.



Now let $\vec{F}$ be a vector field on $\mathbb{R}^3$ and let C be an oriented curve (direction matters!) in $\mathbb{R}^3$ parametrized by

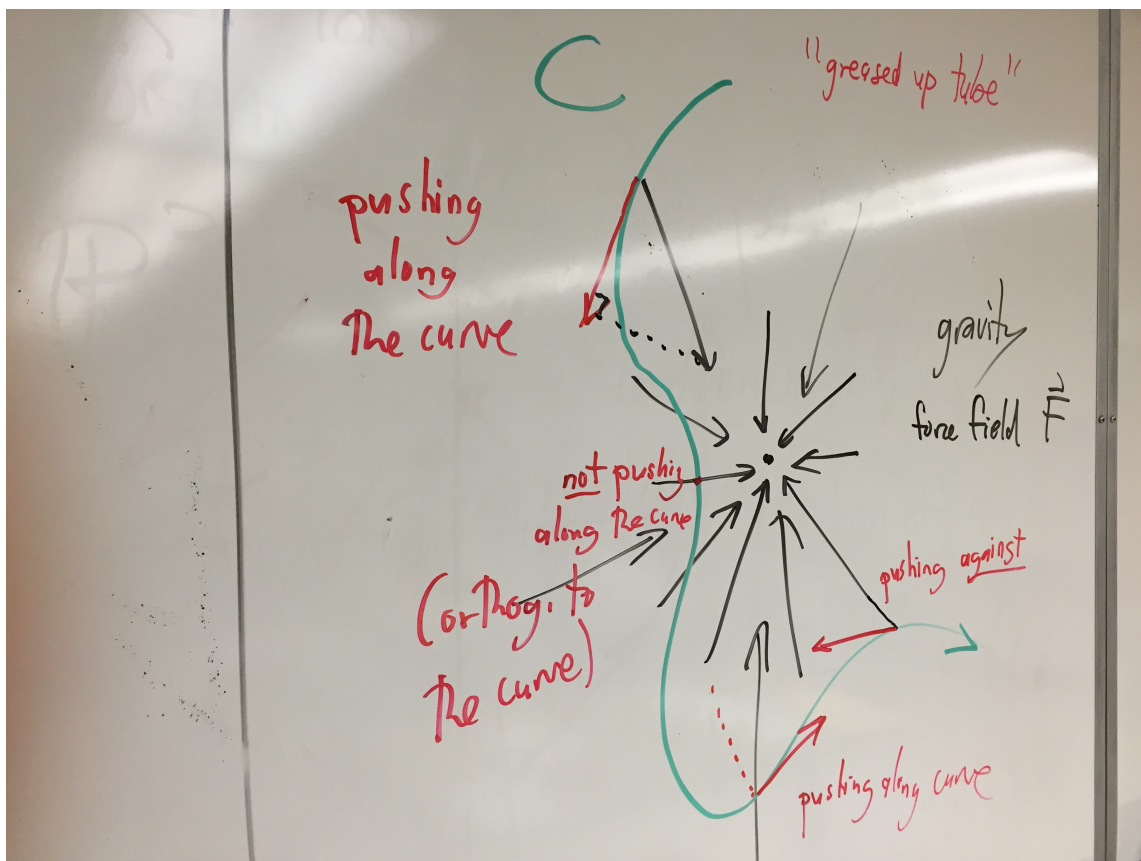
$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle, \quad a \leq t \leq b.$$

Then the component of $\vec{F}$ in the direction of $\vec{r}'(t)$ (that is, in the direction tangent to the curve) at $\vec{r}(t)$ is:

$$(**) \quad \frac{\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)}{\|\vec{r}'(t)\|}.$$

In physics, $\vec{F}$ might represent a *force field*; then $(**)$ represents “how much $\vec{F}$ is pushing along C at the point $\vec{r}(t)$. ”

In some places, $\vec{F}$ might push *forwards* (a positive amount) along the curve. In other places, $\vec{F}$ might push *backwards* (a negative amount) along the curve. In other places, $\vec{F}$ might push *orthogonal* to the curve (hence pushing a *zero* amount along the curve).



Imagine yourself enclosed inside a greased-up glass tube in outer space. If the gravitational force is orthogonal to the tube, it will pull you against the glass. Otherwise, the gravitational force will have some component along the tube, forcing you to slide in that direction.



Question. What is the total (net) amount of work that $\vec{F}$ does, in pushing along C ?

Definition. The *line integral of $\vec{F}$ along C* is defined to be

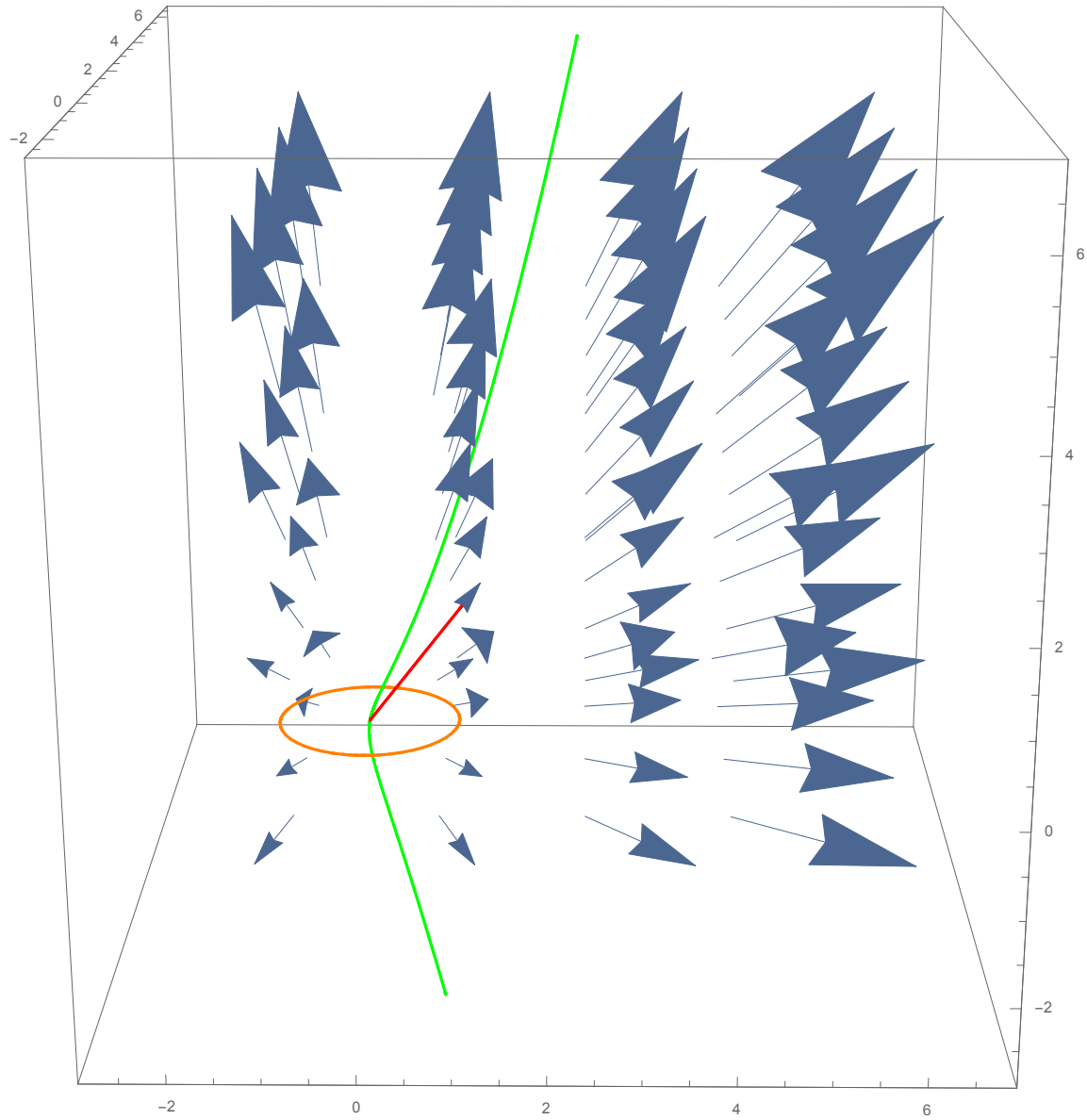
$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &\stackrel{\text{def.}}{=} \int_a^b (\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)) dt \\ &= \int_a^b \left(\frac{\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)}{\|\vec{r}'(t)\|} \right) \|\vec{r}'(t)\| dt. \end{aligned}$$

We wrote it the second way to emphasize that it is the line integral with respect to arclength of the *component of $\vec{F}$ in the direction of the curve*. (See §1 of these notes.) In physics, it represents the work done by $\vec{F}$ along the curve.

Example. Let $\vec{F}(x, y, z) = \langle x, y, z \rangle$. Evaluate the integral of $\vec{F}$ along each of the following curves:

- (a) $\vec{r}(t) = \langle t, t, t \rangle$, $0 \leq t \leq 1$.
- (b) $\vec{r}(t) = \langle \cos t, \sin t, 0 \rangle$, $0 \leq t \leq 2\pi$.

(c) $\vec{r}(t) = \langle t^2, 3t, 2t^3 \rangle, -1 \leq t \leq 2$.



[It should really go all the way up to $z = 16$, but I decided to cut off the picture.]

Hints for each curve:

- (a) The curve is going exactly along the flow (of $\vec{F}$), but $\vec{F}$ is weak near the origin, so the total work of $\vec{F}$ along the curve is positive and approximately 2 (?).
- (b) The curve is always orthogonal to the flow, so $\vec{F}$ does not *ever* push along the curve. The answer is 0.
- (c) The curve starts off going against the flow (but $\vec{F}$ is weak there). But then the curve goes along the flow when $\vec{F}$ is strong. So the total amount of work of $\vec{F}$ along the curve is positive and LARGE.

Solution. In all cases, we need to calculate

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b (\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)) dt.$$

- (a) We have $\vec{r}(t) = \langle t, t, t \rangle$, so $\vec{r}'(t) = \langle 1, 1, 1 \rangle$, and

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = 3t,$$

Thus

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \int_0^1 3t dt \\ &= \frac{3}{2}. \end{aligned}$$

- (b) For this curve, we have $\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = 0$ for all t , so

$$\int_C \vec{F} \cdot d\vec{r} = 0.$$

- (c) For this curve, we have $\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = 2t^3 + 9t + 12t^5$, so

$$\int_C \vec{F} \cdot d\vec{r} = 147.$$

Next up: *conservative* vector fields and then Green's theorem!
(Please come to class.)

Problems taken from Marsden and Weinstein's *Calculus, III*. The text is available free online: <http://authors.library.caltech.edu/25043/>

For the following problems, evaluate the line integral of the given function along the given curve.

$$(1) \quad f(x, y, z) = x + y + yz, \text{ where } \vec{r}(t) = \langle \sin t, \cos t, t \rangle, 0 \leq t \leq 2\pi.$$

$$(2) \quad f(x, y, z) = x + \cos^2 z, \text{ where } \vec{r}(t) = \langle \sin t, \cos t, t \rangle, 0 \leq t \leq 2\pi.$$

$$(3) \quad f(x, y, z) = x \cos z, \text{ where } \vec{r}(t) = \langle t, t^2, 0 \rangle, 0 \leq t \leq 1.$$

$$(4) \quad f(x, y, z) = e^{\sqrt{z}}, \vec{r}(t) = \langle 1, 2, t^2 \rangle, 0 \leq t \leq 1.$$

$$(5) \quad f(x, y, z) = yz, \text{ where } \vec{r}(t) = \langle t, 3t, 2t \rangle, 1 \leq t \leq 3.$$

$$(6) \quad f(x, y, z) = (x + y)/(y + z), \text{ where } \vec{r}(t) = \langle t, \frac{2}{3}t^{3/2}, t \rangle, 1 \leq t \leq 2.$$

For the following problems, evaluate the integral of the given vector field $\vec{F}$ along the given path.

$$(7) \quad \vec{r}(t) = \langle \sin t, \cos t, t \rangle, 0 \leq t \leq 2\pi, \vec{F}(x, y, z) = \langle x, y, z \rangle.$$

$$(8) \quad \vec{r}(t) = \langle t, t, t \rangle, 0 \leq t \leq 1, \vec{F}(x, y, z) = \langle x, -y, z \rangle.$$

$$(9) \quad \vec{r}(t) = \langle \cos t, \sin t, 0 \rangle, 0 \leq t \leq \pi/2, \vec{F}(x, y, z) = \langle x, -y, z \rangle.$$

$$(10) \quad \vec{r}(t) = \langle \cos t, \sin t, 0 \rangle, 0 \leq t \leq \pi/2, \vec{F}(x, y, z) = \langle x, -y, 2 \rangle.$$

$$(11) \quad \vec{r}(t) = \langle \sin t, t^2, t \rangle, 0 \leq t \leq 2\pi, \vec{F}(x, y, z) = \langle \sin z, \cos \sqrt{y}, x^3 \rangle.$$

$$(12) \quad \vec{r}(t) = \langle \cos t, \sec t, \tan t \rangle, \quad -\pi/4 \leq t \leq \pi/4, \quad \vec{F}(x, y, z) = \langle xz, xy, yz \rangle.$$

$$(13) \quad \vec{r}(t) = \langle (1+t^2)^2, 1, t \rangle, \quad 0 \leq t \leq 1, \quad \vec{F}(x, y, z) = \langle 1/(z^2+1), x(1+y^2), e^y \rangle.$$

$$(14) \quad \vec{r}(t) = \langle 3t, t-1, t^2 \rangle, \quad 0 \leq t \leq 1, \quad \vec{F}(x, y, z) = \langle x^2 + x, \frac{x-y}{x+y}, z - z^3 \rangle.$$

Let $\vec{F}(x, y, z) = \langle x^2, -xy, 1 \rangle$. Evaluate the line integral of $\vec{F}$ along each of the curves in Exercises 15–18.

(15) The straight line joining $(0, 0, 0)$ to $(1, 1, 1)$.

(16) The circle of radius 1, centered at the origin and lying in the yz plane, traversed counterclockwise as viewed from the positive x -axis.

(17) The parabola $z = x^2$, $y = 0$, between $(-1, 0, 1)$ and $(1, 0, 1)$.

(18) The straight line between $(-1, 0, 1)$ and $(1, 0, 1)$.

And remember that you can find even *more* practice problems by looking in *other* calculus books.