THE PHYSICAL APPLICATIONS

OF

HYPERGEOMETRIC FUNCTIONS

by

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Introduction

The hypergeometric function ${}_2F_1(a,b;c;z)$ is considered to be a function rarely used in physics and engineering. It would be desirable to know to what uses it is put in these fields. The properties of the hypergeometric function are discussed at some length in the standard treatises on the theory of functions, and it is of important theoretical interest in mathematics. But is it important for the physicist and engineer to be familiar with the function? The answer seems to be yes. The applications of the function are so diverse, that it appears desirable for physicists and electrical engineers to be familiar with the properties of the hypergeometric function in general, and in particular, with those cases in which it can be expressed as simpler functions. In some of the applications herein discussed, it is necessary to notice that in the final result, the hypergeometric function becomes ~~the~~ elliptic function, and numerical answers may be obtained easily. In other examples the hypergeometric series terminates, and so it is necessary to consider Jacobi polynomials. In any case, the fundamental properties of the function should be familiar, and one should be able to recognize the particular cases in which the function can be expressed in a simpler form which is tabulated.

In the following examples, most of the intermediate steps have been removed, with the exception of the examples dealing with the rolling of a hoop, and waves on the surface of a fluid in a circular dish. In the first example, the final solution contains the hypergeometric function in a form which cannot be reduced. In the second example, the hypergeometric series is required to terminate, and this fact determines the allowable frequencies of oscillation of the fluid. In the last example the equation to be solved is Legendre's equation, but the solution is expressed in terms of the hypergeometric function, presumably for reasons of symmetry. In the examples concerned with solving Schrödinger's equation, a factor not evident is that the solutions were selected from the twenty-four possible solutions of the hypergeometric equation. Here it is important for the physicist to be familiar with these solutions, as it enables him to obtain a solution which is finite over the range of the independent variable concerned in the problem.

In the examples dealing with acoustics and with electric currents, the hypergeometric functions arise from the evaluation of integrals of Bessel functions. An integral of particular importance is the Weber-Schaftheitlin integral, and it is used in the acoustical problems and the problem on modulation products. In over half the cases, the hypergeometric function arises from a differential equation, and in the rest, from the evaluation of integrals.

1. As a first example, we will consider the general treatment of the motion of a body, bounded by a surface of revolution, and dynamically symmetrical about the axis of revolution, rolling without sliding on a rough horizontal plane.[n] Let us take as axes, with origin at the center of mass of the body, a set of moving axes turning about themselves with angular velocities p_0, q_0, and r_0, of which the Z axis is the axis of revolution, the Y axis the horizontal in the equator of the body, and the X axis directed toward the ground in the vertical plane containing the Z axis.

We have, for the Euler's angles ϑ, ψ, φ, where here $\varphi = -\pi/2$ hence

$$p_0 = -\psi' \sin\vartheta, \quad q_0 = \vartheta', \quad r_0 = \psi' \cos\vartheta \tag{1.1}$$

where primes denote time derivatives,

These are connected with the rotation of the body by the relation $p_0 = p$, $q_0 = q$, $r_0 = r - \varphi'$.

For the motion of the center of mass, we have the components of the weight of the body

$$X = Mg \sin\vartheta, \qquad Y = 0, \qquad Z = -Mg \cos\vartheta \tag{1.2}$$

together with the unknown components of the reaction, R_x, R_y, R_z. The resultant is to be equated to the product of the mass by the acceleration of the center of mass.

If V_x, V_y, V_z are the components of the velocity of the center of mass along the instantaneous position of the moving axes, we have

$$
\begin{aligned}
M\left(\frac{dV_x}{dt} + q_0 V_x - r_0 V_y\right) &= R_x + Mg \sin\vartheta \\
M\left(\frac{dV_y}{dt} + r_0 V_x - p_0 V_z\right) &= R_y \\
M\left(\frac{dV_z}{dt} + p_0 V_z - q_0 V_x\right) &= R_z - Mg\cos\vartheta
\end{aligned}
\qquad (1.3)
$$

For the rotation we have, since $p=p_0$, $q=q_0$

$$
\begin{aligned}
A\frac{dq}{dt} + (Cr - Ar_0)\,q &= -z R_y \\
A\frac{dp}{dt} - (Cr - Ar_0)\,p &= z R_x - x R_z \\
C\frac{dr}{dt} &= x R_y
\end{aligned}
\qquad (1.4)
$$

where A is the moment of inertia around the X and Y axis, C is the moment of inertia around the Z axis.

We have finally, as the conditions for rolling and pivoting, the equations stating that the velocity at the point of contact with the plane (whose coördinates are x,y,z) is at rest

$$
\begin{aligned}
V_x + q z &= 0 \\
V_y + r x - p z &= 0 \\
V_z - q x &= 0
\end{aligned}
\qquad (1.5)
$$

Let us consider a circular cylinder, where z=c and x=a. a is the radius of the disk or cylinder, and c is the distance of the center of mass from the plane of the circular edge on which the cylinder rolls. After some manipulation we get a differential equation for r in terms of ϑ

$$\frac{d^2 r}{d\vartheta^2} + \cot\vartheta \frac{dr}{d\vartheta} + \frac{Mca(c\cot\vartheta - a)}{AC + M(a^2A + Cc^2)}\, r = 0 \tag{1.6}$$

In the case of the disk c=0, and if we substitute $x = \cos^2\vartheta$ as a new variable we get

$$x(1-x)r'' + \left(\tfrac{1}{2} - \tfrac{3}{2}x\right)r' - \frac{MCa^2}{A(C+Ma^2)}\, r = 0 \tag{1.7}$$

if $\quad \gamma = \frac{1}{2}, \quad \alpha + \beta = \frac{1}{2}, \quad \alpha\beta = \dfrac{MCa^2}{A(C+Ma^2)}$

then

$$\begin{aligned} r &= c_1 F(\alpha, \beta; \gamma; x) + c_2 x^{1-\gamma} F(\alpha-\gamma+1; \beta-\gamma+1; 2-\gamma; x) \\ &= c_1 F(\alpha, \beta; \tfrac{1}{2}; \cos^2\vartheta) + c_2 \cos\vartheta\, F(\alpha+\tfrac{1}{2}, \beta+\tfrac{1}{2}; \tfrac{3}{2}; \cos^2\vartheta) \end{aligned} \tag{1.8}$$

and p and q can be obtained from the original equations.

2. We consider next a hydrodynamical problem. If we have a basin containing a fluid, standing waves can exist on the surface of the fluid.[14] In particular, we will consider a circular basin, the depth of the fluid varying as $1 - r^2/a^2$. From the equation of continuity we have

$$\frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(vh) = -\frac{\partial \zeta}{\partial t} \tag{2.1}$$

where u is the x component of the fluid velocity and v is the y component. ζ is the vertical elevation of the surface above the equilibrium level z_0. The dynamical equations give us

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x}$$

$$\rho \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y} \qquad (2.2)$$

where $p=p_0 + g\rho(z_0 + \zeta - z)$. g is the acceleration due to gravity and ρ is the density of the fluid. From these equations we get

$$\frac{\partial}{\partial x}\left(h \frac{\partial \zeta}{\partial x}\right) + \frac{\partial}{\partial y}\left(h \frac{\partial \zeta}{\partial y}\right) = \frac{1}{g}\frac{\partial^2 \zeta}{\partial t^2} \qquad (2.3)$$

If we transform to polar coordinates, and assume the solution of the form $\zeta = R(r)\Theta(\vartheta)T(t)$ we get

$$T = e^{i(\omega t + \alpha)} \qquad \Theta = \cos(s\vartheta + \beta) \qquad (2.4)$$

andthe equation for R is

$$\frac{r^2}{hR}\left\{h\frac{d^2R}{dr^2} + \frac{h}{r}\frac{dR}{dr} + \frac{dh}{dr}\frac{dR}{dr} + \frac{\omega^2}{g}R\right\} = s^2 \qquad (2.5)$$

assuming the depth h is a function of r only. Substituting our value of the function $h=h_0(1-r^2/a^2)$ we get

$$\left(1-\frac{r^2}{a^2}\right)\left(R'' + \frac{1}{r}R' - \frac{s^2}{r^2}R\right) - \frac{2r}{a^2}R' + \frac{\omega^2}{gh_0}R = 0 \qquad (2.6)$$

where ω is obtained from $T(t)$ and s from $\Theta(\vartheta)$. Writing

$$\frac{\omega^2 a^2}{gh_0} + s^2 = k(k+2)$$

we get as a solution to (2.6)

$$R= a_o(\frac{r}{a})^s \, F(\frac{k+s}{2}; \frac{2+s-k}{2} ; s+1; \frac{r^2}{a^2}). \qquad (2.7)$$

Now the hypergeometric function diverges for r=a, since $(k+s)/2 + (2+s-k)/2 - (s+1) = 0$. Hence we require that the series terminate. This will be the case if $k= s+2m$ where m is an integer. This will determine the values of ω from the relation

$$\frac{\omega^2 a^2}{gh_o} + s^2 = (s+2m)(s+2m -2), \qquad (2.8)$$

and finally the whole solution is

$$= a_o \sin (\omega t+\alpha) \cos (s \vartheta+\beta)$$
$$F(s+m, 1-m; s+1; \frac{r^2}{a^2}). \qquad (2.9)$$

3. If a flexible disk vibrates in a fluid, the effective mass of the disk increases due the the presence of the fluid.[9] In the calculation of this effect, hypergeometric functions are involved, and for this reason, we will consider a thin flexible disk vibrating in a fluid, the disk being situated in an infinite plane. In the first case, we consider a disk with one nodal circle, the edge being free. Assume the dynamic deformation curve to be represented by

$$w= A(1- p_1\frac{r^2}{a^2}) \qquad (3.1)$$

where $A = X \cos \omega t$ and p_1 determines the radius of the nodal circle. The velocity potential ϕ at points on the disk is

$$\phi = \int_0^\infty J_0(kr)\,dk \int_0^a J_0(kr) \frac{\partial w}{\partial t} r\,dr \tag{3.2}$$

where $k = w/c$. Integration of the above equation gives

$$\phi = \dot{A}a\left((1-p_1)F\left(-\tfrac{1}{2}, \tfrac{1}{2}; 1; \frac{r^2}{a^2}\right) + \tfrac{2}{3}p_1F\left(-\tfrac{3}{2}, \tfrac{1}{2}; 1; \frac{r^2}{a^2}\right)\right) \tag{3.3}$$

The kinetic energy of the fluid associated with both sides of the disk is

$$T = \rho \iint \phi \frac{\partial w}{\partial t}\,dS \tag{3.4}$$

and we find that

$$T = 2\pi\rho\dot{A}^2a^3\Big(\tfrac{1}{2}(1-p_1)\big((1-p_1)F(-\tfrac{1}{2};\tfrac{1}{2};2)+\tfrac{1}{2}p_1F(-\tfrac{1}{2};\tfrac{1}{2};3)\big)$$
$$+\tfrac{1}{3}p_1\big((1-p_1)F(-\tfrac{3}{2};\tfrac{1}{2};2)+\tfrac{1}{2}p_1F(-\tfrac{3}{2};\tfrac{1}{2};3)\big)\Big) \tag{3.5}$$

also $T = \tfrac{1}{2}M\dot{A}^2$, so M, the effective mass of the disk is

$$M = (16/3)\rho a^3(1-(14/15)p_1+(5/21)p_1^2) \tag{3.6}$$

If we consider a free edge disk with one nodal diameter, the dynamic deformation curve is of the form

$$w = A\frac{r}{a}\cos\vartheta = \frac{A}{a}x \quad \text{where } x = \cos\vartheta \tag{3.7}$$

then the potential ϕ is determined in the following manner. Assume

$$w = A\left(1 - \frac{r^2}{a^2}\right)^{n+1}$$

then

$$\phi = \dot{A}\int_0^\infty J_0(kr)\,dk \int_0^a J_0(kr)\left(1-\frac{r^2}{a^2}\right)^{n+1} dr \tag{3.8}$$

if we write $r = a\sin\chi$ we have for the first integral of (3.8)

$$a^2\int_0^{\pi/2} J_0(ka\sin\chi)\cos^{2n+3}\chi\sin\chi\,d\chi \tag{3.9}$$

the value of this is

$$2^{n+1}\Gamma(n+2)\,a^2\,\frac{J_{n+2}(ka)}{(ka)^{n+2}}$$

so

$$\phi = 2^{n+1}\Gamma(n+2)\,\dot{A}\,a^2\int_0^\infty \frac{J_0(kr)\,J_{n+2}(ka)}{(ka)^{n+2}}\,dk$$

$$= \frac{\sqrt{\pi}}{2}\,\frac{\Gamma(n+2)}{\Gamma(n+5/2)}\,\dot{A}a\,F\left[-(n+3/2),\,3/2\,;\,1\,;\,r^2/a^2\right] \tag{3.10}$$

To find ϕ for the case $w=A(r/a)\cos\vartheta$ we put $x^2=r^2-y^2$ and

$$\frac{\partial}{\partial x}\left\{A\left(1-r^2/a^2\right)^{n+1}\right\} = -2(n+1)\,\frac{Ax}{a^2}\left(1-r^2/a^2\right)^n \tag{3.11}$$

since $\frac{\partial}{\partial x}F(\alpha,\beta;\gamma;z) = \frac{\alpha\beta}{\gamma}F\left[\alpha+1,\beta+1;\gamma+1;z\right]\frac{\partial z}{\partial x}$

then

$$\frac{\partial\phi}{\partial x} = -\frac{\sqrt{\pi}}{2}\,\frac{\Gamma(n+2)}{\Gamma(n+3/2)}\,\frac{\dot{A}x}{a}\,F\left[-(n+1/2),\,3/2\,;\,2\,;\,r^2/a^2\right] \tag{3.12}$$

If n=0, (3.11) becomes

$$-\frac{2Ax}{a^2} \tag{3.13}$$

and (3.12) becomes

$$-\dot{A}\,x/a\,F\left(-\tfrac{1}{2},\tfrac{3}{2};2;r^2/a^2\right) \tag{3.14}$$

If $w=A(x/a)$ then ϕ is formed if (3.14) is multiplied by $-a/2$ so finally

$$\phi = \frac{\dot{A}}{2}\,x\,F\left(-\tfrac{1}{2},\tfrac{3}{2};2;\frac{r^2}{a^2}\right) \tag{3.15}$$

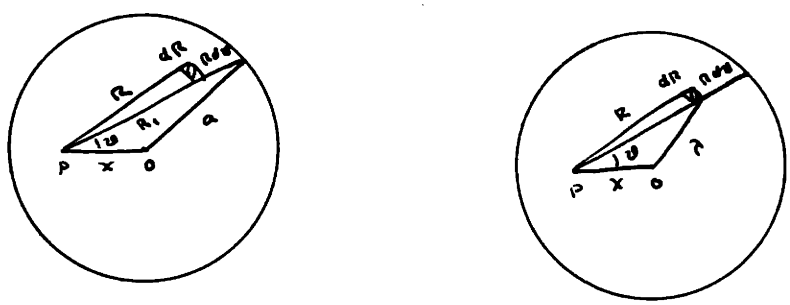
The result for a free edge disk with a stationary center is obtained by using $w=A(r^2/a^2)$ and we find that

$$\phi = \frac{\dot{A}}{a^2}\int_0^\infty J_0(kr)\,dk\int_0^a J_0(kr)\,r^3\,dr$$

$$= \dot{A}a\left\{F(-\tfrac{1}{2},\tfrac{1}{2};1;r^2/a^2)-\tfrac{2}{3}F(-\tfrac{3}{2},\tfrac{1}{2};1;r^2/a^2)\right\} \qquad (3.16)$$

Once ϕ has been obtained, the effective mass is determined in the same manner as in the first of these examples.

4. A somewhat similar example to the preceding is the calculation of the total pressure on a vibrating disk.[10]



The pressure at P due to the area element RdRdθ is given by

$$dp = \frac{i\rho\omega\dot{A}}{2\pi}\,\frac{e^{-ikR}}{R}\,R\,dR\,d\vartheta \qquad (4.1)$$

where ρ = density of the fluid; $k=\omega/c=2\pi/\lambda$; $\dot{A}$ is the axial velocity and ω is the angular frequency of vibration. The total pressure at P is

$$p = \frac{i\rho c\dot{A}}{\pi}\int_0^\pi d\vartheta\int_0^{R_1} k\,e^{-ikR}\,dR \qquad (4.2)$$

we have $R_1^2 - 2R_1x\cos\theta + (x^2-a^2) = 0$, or

$$R_1 = x\cos\theta \pm a(1-b^2\sin^2\theta)^{\frac{1}{2}} = \mu \pm \chi \qquad (4.3)$$

where $b=x/a$. Now the first integral of (4.2) is

$$k\int_0^{\mu+\chi} e^{-ikR}\,dR = i\left[e^{-ik(\mu+\chi)} - 1\right]$$

Then the second integral becomes

$$\int_0^{\pi}\Big[\Big\{k(\mu+\chi)-\frac{k^3}{3!}(\mu+\chi)^3+\cdots\Big\}$$
$$-i\Big\{\frac{k^2}{2!}(\mu+\chi)^2-\frac{k^4}{4!}(\mu+\chi)^4+\cdots\Big\}\Big]d\vartheta \qquad (4.4)$$

now

$$k\int_0^{\pi}(\mu+\chi)\,d\vartheta=\pi z\,F(-\tfrac{1}{2},\tfrac{1}{2};1;b^2) \qquad (4.5)$$

where z=ka, and similarly for the other odd powers of $\mu+\chi$. The even powers are polynomials. Finally p is given by

$$p=\rho c\dot{A}\Big\{\frac{z^2}{2!}f_2-\frac{z^4}{4!}f_4+\cdots+i\Big(zf_1-\frac{z^3}{3!}f_3+\cdots\Big)\Big\} \qquad (4.6)$$

where

$$f_2=1$$
$$f_4=1+2b^2$$
$$-\ -\ -\ -\ -\ -$$
$$f_1=F(-\tfrac{1}{2},\tfrac{1}{2};1;b^2) \qquad (4.7)$$
$$f_3=F(-\tfrac{3}{2},\tfrac{1}{2};1;b^2)+\tfrac{3}{2}b^2F(-\tfrac{1}{2},\tfrac{1}{2};2;b^2)$$

In case the disk is flexible, the pressure p becomes

$$p=\rho c\dot{A}\Big\{\frac{z^2}{2!}g_2-\frac{z^4}{4!}g_4+\cdots\cdots$$
$$+i\Big(zg_1-\frac{z^3}{3!}g_3+\frac{z^5}{5!}g_5-\cdots\cdots\Big)\Big\} \qquad (4.8)$$

where

$$g_2 = 1 - p_1/2$$
$$g_4 = 1 + 2b^2 - p_1(2/3 + b^2)$$
$$- \; - \; - \; - \; - \; - \; - \; - \qquad (4.9)$$
$$g_1 = F(-\tfrac{1}{2}, \tfrac{1}{2}; 1; b^2) - p_1\Big\{\tfrac{1}{3}F(-\tfrac{3}{2}, \tfrac{1}{2}; 1; b^2)$$
$$+ b^2\big[F(-\tfrac{1}{2}, \tfrac{1}{2}; 1; b^2) - \tfrac{1}{2}F(-\tfrac{1}{2}, \tfrac{1}{2}; 2; b^2)\big]\Big\}$$

the p_1 originates in the formula for the dynamic deformation curve $w=A(1-p_1(r^2/a^2))$ which was assumed for the flexible disk.

5. For the next example, we will consider the output voltage of a rectifier when two input frequencies are applied to the input.[8] In the first instance, we assume the rectifier is a half wave linear one, with a transfer characteristic

$$E(t)=e(t) \qquad e(t) > 0$$
$$=0 \qquad e(t) \leqslant 0 \qquad (5.1)$$

the input voltage is

$$e(t) = P\cos(pt+\vartheta_p) + Q\cos(qt+\vartheta_q) \qquad (5.2)$$

hence the output voltage is

$$E(t) = f(x,y) = P(\cos x + k\cos y) \qquad \cos x + k\cos y > 0$$
$$= 0 \qquad \cos x + k\cos y \leqslant 0 \qquad (5.3)$$

where we consider P 0 and $k=Q/P$. It is to be noted that $f(x,y)$ is periodic with period 2π in x and y, so we can expand it in a double Fourier series:

$$f(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[A_{\pm n,m}\cos(mx \pm ny) + B_{\pm m,n}\sin(mx \pm ny)\right] \qquad (5.4)$$

The coefficients are determined by the formulae

$$A_{\pm m,n} = \frac{1}{2\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\cos(mx \pm ny)\,dy\,dx$$

$$B_{\pm m,n} = \frac{1}{2\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\sin(mx \pm ny)\,dy\,dx \qquad (5.5)$$

In our problem we let

$$\begin{aligned} x &= pt + \vartheta_p \\ y &= qt + \vartheta_q \end{aligned} \qquad (5.6)$$

and the values of x and y are obtained by eliminating t, yielding

$$y = (q/p)x + \vartheta_q - (q/p)\vartheta_p \qquad (5.7)$$

Since the representation (5.4) holds everywhere in the x,y plane, it is valid along the line given by (5.7). Hence the equations (5.5) will give the amplitudes of the frequencies $(mp \pm nq)/2\pi$. The phase angles are given by $m\vartheta_p \pm n\vartheta_q$ corresponding to the above frequencies.

To integrate (5.5) we can consider the following relation

$$\begin{aligned} \frac{u}{2} + \frac{u}{\pi}\int_0^\infty \frac{\sin u\lambda}{\lambda}\,d\lambda &= u \qquad u \geq 0 \\ &= 0 \qquad u < 0 \end{aligned} \qquad (5.8)$$

if we substitute u= P cos x + G cos y we get

$$f(x,y) = \frac{u}{2} + \frac{u}{\pi}\int_0^\infty \frac{\sin \lambda u}{\lambda}\, d\lambda$$

using the above relation in (5.5) it can be deduced that

$$A_{m,n} = \frac{2}{\pi}(-1)^{\frac{m+n+2}{2}} \int_0^\infty \frac{J_m(\lambda P)\, J_n(\lambda Q)}{\lambda^2}\, d\lambda \tag{5.9}$$

where m+n is even and greater than zero. This is a special case of the Weber and Schafheitlin Integral. If m+n=0 then (5.9) must be replaced by a contour integral taken along the real axis with an indentation at the origin. All other quantities remain the same except for a factor of one-half. Then it may be shown that

$$A_{m,n} = \frac{(-1)^{\frac{m+n+2}{2}}\,\Gamma\left(\frac{m+n-1}{2}\right) k^n P}{2\pi\,\Gamma(n+1)\,\Gamma\left(\frac{m-n+3}{2}\right)} F\left(\frac{m+n-1}{2}, \frac{n-m-1}{2}; n+1; k^2\right) \tag{5.10}$$

for $m+n=2\nu$ where $\nu = 0,1,2,3, \cdots$. By other considerations, if m+n is odd and greater than one, $A_{mn}=0$. If m+n=1, $A_{10} = P/2$, $A_{01} = Q/2$.

If we consider a half wave square law detector or rectifier, we have

$$f(x,y) = P^2(\cos x + k\cos y)^2 \qquad \cos x + k\cos y > 0$$

$$= 0 \qquad \cos x + k\cos y < 0$$

and it can be shown as before that

$$A_{mn} = \frac{(-1)^{\frac{m+n+1}{2}} k^n p^2 \Gamma\left(\frac{m+n-2}{2}\right)}{2\pi \Gamma(n+1) \Gamma\left(\frac{m-n+4}{2}\right)} F\left(\frac{m+n-2}{2}, \frac{n-m-2}{2}; n+1; k^2\right) \quad (5.11)$$

for odd order products. For all even order products greater than two, $A_{mn}=0$, and $A_{00}= p^2(1+k^2)/4$; $A_{20}= P^2/4$; $A_{11}= P^2k/2$ and $A_{02}= P^2k^2/4$.

6. In the mathematical analysis of random noise,[6,7] it is necessary to evaluate certain multuple integrals, the following of which are expressible in terms of the hypergeometric function

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, x^n y^m \exp(-x^2-y^2-2xy\cos\varphi) \qquad (6.1)$$

$= 0$ if n+m is odd

$$= (-1)^n \sqrt{\pi} \frac{\Gamma\left(\frac{n+m+1}{2}\right)}{(\sin\varphi)^{n+m+1}} F\left(-n, -m; \frac{1-n-m}{2}; \frac{1-\cos\varphi}{2}\right)$$

if n+m is even

If several frequencies plus noise are passed through filters or detectors, the envelope of the output wave R can be found by elaborate analysis. Among the results obtained are asymptotic expansions for $\bar{R}_n$ these expansion are valid for large signal to noise ratios. If the input voltage V is $V=V_n + P \cos pt + Q \cos qt$, then

$$I_{dc} = \overline{I}_{ef} \sim \frac{\alpha}{\pi} P \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})_k (-\frac{1}{2})_k}{k!\ x^k} {}_2F_1\left[(k-\tfrac{1}{2}),\ k-\tfrac{1}{2};\ 1;\ \eta/x\right]$$

$$x = P^2/2\psi_0 \qquad \eta = Q^2/2\psi_0 \tag{6.2}$$

where $\alpha I = V \qquad V > 0$

and ψ_0 is the mean-square value of V_n.

7. We turn next to the solution of a wave mechanics problem.[1] Several solutions of Schrödinger's equation for a potential barrier have been obtained, mostly for the case of a barrier having discontinuous derivatives. In this example, we will consider a smooth potential function given by

$$V(x) = -A\frac{\xi}{1-\xi} - B\frac{\xi}{(1-\xi)^2}$$
$$\xi = -e^{2\pi x/l} \tag{7.1}$$

This function is analytic and possesses an extremum whose maximum value is

$$V(x_m) = V_m = \frac{(A+B)^2}{4B} \tag{7.2}$$

if $B \geqslant 0$. The wave equation of an electron moving under the action of this potential is

$$\frac{d^2u}{dx^2} + \frac{8\pi^2 m}{h^2}\left\{\frac{A\xi}{1-\xi} + \frac{B\xi}{(1-\xi)^2} + W\right\}u = 0 \tag{7.3}$$

or

$$\xi^2 u'' + \xi u' + \frac{2m\ell^2}{h^2}\left\{ \frac{A\,\xi}{1-\xi} + \frac{B\,\xi}{(1-\xi)^2} + W \right\} u = 0 \tag{7.4}$$

where primes denote differentiation with respect to ξ. This equation is of the hypergeometric type, and the solution applicable here is

$$u = (1-\xi)^{i\beta}\left(\frac{\xi}{\xi-1}\right)^{i\alpha} F\left(\tfrac{1}{2}+i(\alpha-\beta+\delta), -\tfrac{1}{2}+i(\alpha-\beta-\delta); 1-2i\beta; \frac{1}{1-\xi}\right) \tag{7.5a}$$

when $x > 1$, and

$$\begin{aligned} u = {} & a_1\left(\frac{\xi}{\xi-1}\right)^{i\alpha}(1-\xi)^{i\beta} F\left(\tfrac{1}{2}+i(\alpha-\beta+\delta), -\tfrac{1}{2}+i(\alpha-\beta-\delta); 1+2i\alpha; \frac{\xi}{\xi-1}\right) \\ & + a_2\left(\frac{\xi}{\xi-1}\right)^{-i\alpha}(1-\xi)^{i\beta} F\left(\tfrac{1}{2}+i(-\alpha-\beta+\delta), -\tfrac{1}{2}+i(-\alpha-\beta-\delta); 1-2i\alpha; \frac{\xi}{\xi-1}\right) \end{aligned} \tag{7.5b}$$

otherwise.

When x is large and negative, we should have a DeBroglie wave, the wavelength of which is $\lambda = h/(2mW)^{\frac{1}{2}}$, and when x is large and positive, a wave of wavelength $\lambda' = h/(2m(W-A))^{\frac{1}{2}}$. The two waves (traveling in opposite directions) for $x \ll 0$ should be of the form

$$a_1 e^{\frac{2\pi i x}{\lambda}} + a_2 e^{-\frac{2\pi i x}{\lambda}} = a_1(-\xi)^{i\alpha} + a_2(-\xi)^{-i\alpha} \qquad \alpha = \frac{\ell}{\lambda} \tag{7.6}$$

and for $x \gg 0$ the wave is

$$e^{\frac{2\pi i x}{\lambda'}} = (-\xi)^{i\beta} \qquad \beta = \ell/\lambda' \tag{7.7}$$

we call C the energy of an electron whose DeBroglie wavelength is 2ℓ, the width of the region of varying potential: $\sqrt{2mC} = h/2\ell$

then we define $\delta = \frac{1}{2}\left(\frac{B-C}{C}\right)^{1/2}$

and we get the important result that the reflection coefficient of the barrier ρ is

$$\rho = \left| \frac{\Gamma(\frac{1}{2}+i(\delta-\beta-\alpha))\,\Gamma(\frac{1}{2}+i(-\delta-\beta-\alpha))}{\Gamma(\frac{1}{2}+i(\delta-\beta+\alpha))\,\Gamma(\frac{1}{2}+i(-\delta-\beta+\alpha))} \right|^2 \tag{7.8}$$

Here is a case in which the knowledge of the twenty-four solutions of the hypergeometric equation enables us to get the proper asymptotic behavior of the wave function.

8. We consider next, a different potential distribution which is used in the theory of vibrations of polyatomic molecules[3]. The distribution is

$$V(x) = B \tanh\left(\frac{x}{d}\right) - C \operatorname{sech}^2\left(\frac{x}{d}\right) \tag{8.1}$$

If $|B| < 2C$ then there is a minimum value at

$$x_0 = -\tanh^{-1}\left(\frac{B}{2C}\right) \tag{8.2}$$

and

$$V(x_0) = -\frac{4C^2+B^2}{4C} \tag{8.3}$$

Using this potential distribution, the wave equation is

$$\frac{d^2\psi}{dz^2} + (-\epsilon - \beta \tanh z + \gamma \operatorname{sech}^2 z)\,\psi = 0 \tag{8.4}$$

where $z = x/d$, $\epsilon = -\frac{8\pi^2 M d^2}{h^2} E$

$$\beta = \frac{8\pi^2 M d^2}{h^2} B \qquad \gamma = \frac{8\pi^2 M d^2}{h^2} C$$

and E is the allowed energy of the system.

If we call

$$\psi = e^{az} \cosh^{-b} z \; F(z) \tag{8.5}$$

we get

$$F'' + 2(a - b\tanh z)F' + [\gamma - b(b+1)]\operatorname{sech}^2 z \, F = 0 \tag{8.6}$$

if $$a = -\tfrac{1}{2}\left[(\epsilon+\beta)^{1/2} - (\epsilon-\beta)^{1/2}\right] \qquad (\epsilon+\beta)^{1/2} > 0$$

and $$b = \tfrac{1}{2}\left[(\epsilon+\beta)^{1/2} + (\epsilon-\beta)^{1/2}\right] \qquad (\epsilon-\beta)^{1/2} > 0$$

then $\psi/F < \infty$ for $-\infty \leq x \leq \infty$.

Let u= (1 + tanh z)/2, then

$$u(1-u)F'' + [a+b+1-2(b+1)u]F' + [\gamma - b(b+1)]F = 0 \tag{8.7}$$

and a solution of the above equation, finite at the origin is

$$F = F\left(b+\tfrac{1}{2}-(\gamma+\tfrac{1}{4})^{1/2},\ b+\tfrac{1}{2}+(\gamma+\tfrac{1}{4})^{1/2};\ a+b+1;\ u\right) \tag{8.8}$$

In order that ψ be finite everywhere it is necessary that $$b = (\gamma+\tfrac{1}{4})^{1/2} - n - \tfrac{1}{2}$$

where n is an integer. In this case F becomes a Jacobi polynomial. Solving for a we have

$$a = -\beta/\left[(4\gamma+1)^{1/2} - 2n - 1\right]$$

so the allowed values of the energy E are

$$E_n = \frac{h^2}{8\pi^2 M d^2}(a^2+b^2)$$

$$= \frac{1}{4}\left[(4c+g^2)^{1/2} - g(2n+1)\right]^2 + \frac{B^2}{\left[(4c+g^2)^{1/2} - g(2n+1)\right]^2} \tag{8.9}$$

here

$$0 \leq n \leq (\gamma+\tfrac{1}{4})^{1/2} - (\beta/2)^{1/2} - \tfrac{1}{2}$$

$$g^2 = h^2/8\pi^2 M d^2$$

Finally

$$\psi_n = N_n e^{\frac{ax}{d}} \cosh^{-b}(x/d)\, F\left(-n,\ (4\gamma+1)^{1/2}-n;\ a+b+1;\ \tfrac{1}{2}[1+\tanh x/d]\right) \tag{8.10}$$

where N_n is the normalizing constant.

9. As the last example we consider a solution to an equation due to Eddington for the darkening of a spectral line due to absorption.[11] In the case where

the ratio of line absorption to continuous absorption η is connected by the following relation to the optical depth τ,

$$\frac{1}{1+\eta} = \alpha + \beta\tau + \gamma\tau^2 \tag{9.1}$$

the equation for the intensity within the line J'_ν is

$$(\alpha+\beta\tau+\gamma\tau^2)\frac{d^2 J'_\nu}{d\tau^2} + (\beta+2\gamma\tau)\frac{dJ'_\nu}{d\tau} - 3J'_\nu = 3H(2+3\tau) \tag{9.2}$$

The general solution is obtained by letting

$$z = \frac{\tau}{c} + \frac{\beta}{2\gamma c} \qquad n(n+1) = 3/\gamma$$

$$c^2 = \frac{(\beta^2 - 4\alpha\gamma)}{4\gamma^2}$$

so we get Legendre's equation

$$(1-z^2)\frac{d^2 J'_\nu}{dz^2} - 2z\frac{dJ'_\nu}{dz} + n(n+1)J'_\nu = 0 \tag{9.3}$$

and the solutions desired are

$$\begin{aligned} J'_\nu &= a + b\tau + A\,F\left(-n, n+1; 1; \tfrac{1+z}{2}\right) \\ &\quad + B\,F\left(-n, n+1; 1; \tfrac{1-z}{2}\right) \\ &= a + b\tau + A\,F_1(z) + B\,F_2(z) \end{aligned} \tag{9.4}$$

where $a + b\tau$ is the particular solution. Considerations of the values of α, β, γ used in practice leads to the conclusion that the final result is of the form

$$\frac{H'_\nu}{H} = \frac{1}{2}\left\{a + A\,F_1(z_0) + B\,F_2(z_0)\right\} \tag{9.5}$$

where H'_ν/H gives the darkening of the line, and z_0 is the value of z when $\tau=0$.

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