Conditions for chainability of inverse limits on $[0, 1]$ with interval-valued functions

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Abstract

For an inverse sequence on $[0, 1]$ with interval-valued functions, we establish necessary conditions on the bonding functions for chainability of the inverse limit space. We also characterize chainability of the inverse limit in this setting in terms of properties of the bonding functions f_i and the induced functions $F_n\colon [0, 1] \to G'(f_1, \ldots, f_{n-1})$. The properties, in both cases, are related to how triods arise in the partial graphs associated with the inverse sequence when each graph $G(f_i)$ is chainable.

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1. Introduction

In the setting of inverse limits on $[0, 1]$ with interval-valued functions, we determine conditions on the bonding functions that are necessary for chainability of the inverse limit space. Sufficient conditions for chainability in this setting, related to C-sets in the graphs of the bonding functions, were established by W.T. Ingram and the author in [5]. We also provide sufficient conditions on the bonding functions for chainability of the inverse limit. Our conditions are related to the C-set notion, but are not as restrictive as those in [5]. Furthermore, we establish related conditions, on the bonding functions f_i and on the induced continuum-valued functions $F_n\colon [0, 1] \to G'(f_1, \ldots, f_{n-1})$, that characterize chainability of the inverse limit. In our setting, it has been

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shown that if the inverse limit X is tree-like, or equivalently one dimensional, then X is chainable if and only if it is atriodic if and only if all partial graphs (Mahavier products) in the inverse sequence are chainable [9, Corollary 2]. Hence, the conditions mentioned above are ones that determine how triods are formed in the partial graphs when the graphs of the individual bonding functions are chainable. It follows from Theorem 1 in [9] that if a partial graph in our setting contains a triod, then it contains either a one-sided or two-sided triod. So, these special triods are defined, relative to our setting, in section 3. They play a critical role in establishing the results in this paper.

In Section 6, we provide examples that illustrate how our results easily determine chainability or non-chainability of inverse limits on $[0,1]$ with interval-valued functions.

2. Basic definitions

A *compactum* is a compact metric space. A *continuum* is a connected compactum. A *mapping* is a continuous function. A continuum X is *chainable* if for each $\epsilon > 0$, X admits a finite ϵ-chain of open sets covering X. A continuum X is *arclike* if for each $\epsilon > 0$, X admits an ϵ-mapping onto $[0,1]$. It is well-known that for a continuum X, the following are equivalent.

- (i) X is chainable.
- (ii) X is arclike.
- (iii) X is representable as an inverse limit of an inverse sequence on $[0,1]$ with mappings for bonding functions.

See the end of Section 2 in [2] for specific definitions and discussion of these equivalences.

Let X and Y be compacta. We refer to functions $f\colon X \to 2^Y$ as *set-valued functions* from X to Y and we write $f\colon X \to Y$ is a set-valued function. Note that throughout, we are assuming that, for $x \in X$, the value $f(x)$ of a set-valued function is a closed set. The *graph* of f, which we denote by $G(f)$, is the set in $X \times Y$ consisting of all points (x, y) with $y \in f(x)$.

A set-valued function $f\colon X \to Y$ is *upper semi-continuous at the point* $x \in X$ if for each open set V in Y containing the closed set $f(x)$, there is an open set U in X such that $x \in U$, and $f(p) \subset V$ for each $p \in U$. If $f\colon X \to Y$ is upper semi-continuous at each point of X, then f is said to be *upper semi-continuous*. All set-valued functions considered in this paper will be upper semi-continuous.

The set-valued function $f\colon X \to Y$ is *surjective* if for each $y \in Y$, there exists $x \in X$ such that $y \in f(x)$. A set-valued function $f\colon X \to Y$ is *continuum-valued* if for each $x \in X$, the set $f(x)$ is a subcontinuum of Y. If $x \in X$ and $f(x)$ is degenerate, we will sometimes treat $f(x)$ as a point of Y. For $f\colon X \to Y$ a set-valued function, and $A \subset X$, we let $f|_A$ be the set-valued function whose domain is A, and $f|_A(x) = f(x)$ for $x \in A$.

For $i \geq 1$, let X_i be a compactum, and let $X_i \xleftarrow{f_i} X_{i+1}$ be a surjective, set-valued function. Throughout, we let $\{X_i, f_i\}$ denote an inverse sequence, and its inverse limit is given by

$$\varprojlim\{X_i, f_i\} = \{x = (x_1, x_2, \ldots) \in \prod_{i \geq 1} X_i \mid x_i \in f_i(x_{i+1}) \text{ for } i \geq 1\}.$$

For $1 \leq j \leq n$, we define the set below.

$$G_j^{n+1} = G'(f_j, \ldots, f_n) = \{x \in \prod_{i=j}^{n+1} X_i \mid x_i \in f_i(x_{i+1}) \text{ for } j \leq i \leq n\}.$$

We refer to these sets as *partial graphs* in the inverse sequence. For consistency of notation, we let $G_1^1 = X_1$. We point out that some authors use the notation $\star_{i \in [1,n]} \Gamma(f_i)$ for the sets G_1^{n+1}, and call them Mahavier products.

For each $n \geq 1$, let $G_1^n \xleftarrow{F_n} X_{n+1}$ be the set-valued function such that $(x_1, \ldots, x_n)$ is in $F_n(x_{n+1})$ if and only if $(x_1, x_2, \ldots, x_n, x_{n+1})$ is in G_1^{n+1}. We refer to F_n as an *induced function in the inverse sequence*. We note that the domain of F_n is the $(n+1)^{\text{th}}$ factor space in the inverse sequence, and the range of F_n lies in $\prod_{i=1}^n X_i$. So, all notation related to the graphs of the bonding functions f_i and the induced functions F_n, in an inverse sequence, will be relative to the ordering in the inverse sequence. Hence, under this convention, $G(F_n) = G_1^{n+1}$. V. Nall introduced the function F_n in [11], and showed that F_n is upper semi-continuous. If f_i is continuum-valued for each $1 \leq i \leq n$, it was shown in [7] that F_n is continuum-valued.

For infinite inverse sequences considered in this paper, we assume throughout that, for each $i \geq 1$, $X_i = [0,1]$, and $[0,1] \xleftarrow{f_i} [0,1]$ is interval-valued. Again, we assume the domain and range of each f_i are, respectively, the $(i+1)^{\text{th}}$ factor space and the i^{th} factor space. For such inverse sequences, it follows from [4, Theorems 4.1, 4.3, & 4.7] that the inverse limit is a continuum, and, for each $n \geq 1$, $G(f_n)$ and G_1^{n+1} are continua.

An *interval* is a subcontinuum of $[0,1]$. For $a, b \in [0,1]$, we let $[a,b]$ denote the minimal interval containing a and b. Unless specified, there is no assumption that $a \leq b$.

A continuum is *decomposable* if it is the union of two proper subcontinua. A nondegenerate continuum is *indecomposable* if it is not decomposable. If each subcontinuum of X is decomposable, then X is *hereditarily decomposable.* A continuum is *hereditarily unicoherent* if the intersection of each pair of its subcontinua is connected. A continuum is a λ-*dendroid* if it is hereditarily unicoherent and hereditarily decomposable.

A continuum X is a *triod* if there exists a subcontinuum Z of X such that $X \setminus Z$ is the union of three nonempty sets, each two of which are disjoint. A unicoherent continuum T is a triod if there exist three subcontinua J_1, J_2, and J_3 of T such that $T = J_1 \cup J_2 \cup J_3$, $J_1 \cap J_2 \cap J_3 \neq \emptyset$, and for each $i \in \{1,2,3\}$, $J_i \setminus (J_j \cup J_k) \neq \emptyset$ for $\{i,j,k\} = \{1,2,3\}$, see [10, Theorem 11.26]. Throughout, when we need to establish that a continuum T is a triod, we will have assumed conditions that give us that T is unicoherent. So, in these cases, we show that T is a union of three subcontinua satisfying the properties above. Furthermore, if we write $T = J_1 \cup J_2 \cup J_3$ is a triod, we mean that T is unicoherent, and the J_i's satisfy the conditions above. A continuum is *atriodic* if it contains no triod.

Given a set-valued function $f\colon X \to Y$ between compacta, let $c_1\colon G(f) \to X$ and $c_2\colon G(f) \to Y$ denote coordinate projection. If X is a compactum, and $A \subset X$, we let $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ denote, respectively, the closure and the interior of A in X. We denote the covering dimension of X by $\dim(X)$.

3. Definitions related to continuum-valued functions $f\colon [0,1] \to Y$

Let Y be a continuum, and $f\colon [0,1] \to Y$ be a surjective continuum-valued function such that $G(f)$ is hereditarily unicoherent. For such functions f, we define the terminology that follows in this section. For $t \in [0,1]$, we refer to the sets $\{t\} \times f(t)$ as *fibers* of f in $G(f)$.

Definition 1. We say that $G(f)$ contains a *one-sided triod* (at r) if there exists a nondegenerate interval $[r,s]$, two subcontinua A and B of $\{r\} \times f(r)$, and a subcontinuum K of $G(f)$ such that $c_1(K) = [r,s]$ and $T = A \cup B \cup K$ is a triod.

Definition 2. We say that $G(f)$ contains a *two-sided triod* (at t) if there exist an interval $[u, v] \subset [0, 1]$ with $u < t < v$, and subcontinua A and B of $G(f)$ such that $c_1(A) = [u, t]$, $c_1(B) = [t, v]$, and $T = A \cup B \cup (\{t\} \times f(t))$ is a triod.

Both the graphs of continuum-valued functions $f\colon [0, 1] \to Y$, and the partial graphs G_1^{n+1} in inverse sequences on $[0, 1]$ with interval-valued functions admit continuum folder structures. Details of this remark can be found in the first few pages of section 4 in [9]. For graphs of continuum-valued functions, this means that the fibers $\{t\} \times f(t)$ form an upper semi-continuous decomposition of $G(f)$ where the quotient space is $[0, 1]$. Theorem 1 in [9], stated for graphs of continuum-valued functions f, says that if the fibers of f are atriodic and $G(f)$ contains a triod, then $G(f)$ contains either a one-sided or two-sided triod. Theorem 1 in [9] will be useful for determining if graphs of continuum-valued functions are atriodic.

Definition 3. For $t \in [0, 1]$, f is *left cohesive* at t provided either $t = 0$, or $\{t\} \times f(t) \subset \mathrm{cl}(G(f|_{[0,t)}))$. Similarly, f is *right cohesive* at t if either $t = 1$, or $\{t\} \times f(t) \subset \mathrm{cl}(G(f|_{(t,1]}))$. We say that f is *cohesive* at t if it is left cohesive and right cohesive at t. We say that $G(f)$ is *cohesive* if f is cohesive at each $t \in [0, 1]$.

If $I = [r, t]$ is a nondegenerate subinterval of $[0, 1]$, we often say that f is *cohesive on the I side of t* rather than saying f is left or right cohesive at t.

Definition 4. For $t \in [0, 1]$, f is *fully left* (*right*) *cohesive* at t if for each $y \in f(t)$, there exists an increasing (decreasing) sequence $\{t_i\}_{i \geq 1}$ converging to t such that, for each $i \geq 1$, $y \in f(t_i)$.

Clearly, if f is fully left (right) cohesive at t, then f is left (right) cohesive at t. The function ℓ_2 in Figure 3 of section 6 is fully left cohesive at $\frac{1}{2}$, but not fully right cohesive at $\frac{1}{2}$.

A subset H of a continuum X is a *C-set in X* provided that whenever K is a subcontinuum of X that meets both H and $X \setminus H$, we have that $H \subset K$. We note that some authors have analogously defined the phrase H is *terminal in X*. As with the definition of left and right cohesiveness, we define the notions of left side and right side C-sets for fibers of f.

Definition 5. A fiber $\{t\} \times f(t)$ is a *left side C-set in $G(f)$* provided either $t = 0$, or whenever K is a subcontinuum of $G(f|_{[0,t]})$ that meets both $\{t\} \times f(t)$

and $G(f|_{[0,t)})$, we have that $\{t\} \times f(t) \subset K$. A fiber $\{t\} \times f(t)$ is a *right side C-set in $G(f)$* is defined similarly. If each fiber in $G(f)$ is a C-set, we say that f is *C-set-valued*.

We observe the following equivalence of a left (right) cohesive fiber and a left (right) side C-set fiber in certain graphs of continuum-valued functions.

Observation 1. *Suppose $G(f)$ is hereditarily unicoherent and contains no two-sided triod. Let $t \in [0,1]$. Then f is left (right) cohesive at t if and only if $\{t\} \times f(t)$ is a left (right) side C-set in $G(f)$.*

Proof. We prove the equivalence for the left side case.

$\Rightarrow$: Let H be a subcontinuum of $G(f|_{[0,t]})$ such that $c_1(H) = [r,t]$ for some $r < t$. Suppose, for some s with $r < s < t$, $\{s\} \times f(s) \not\subset H$. Then clearly $(H \cap G(f|_{r,s]})) \cup (H \cap G(f|_{s,t]})) \cup (\{s\} \times f(s))$ is a two-sided triod in $G(f)$, contradicting the hypothesis. So, $G(f|_{(r,t)}) \subset H$, and also $\mathrm{cl}(G(f|_{(r,t)})) \subset H$. Since f is left cohesive at t, $\{t\} \times f(t) \subset \mathrm{cl}(G(f|_{(r,t)}))$. We have that $\{t\} \times f(t) \subset H$. Thus, $\{t\} \times f(t)$ is a left side C-set in $G(f)$.

$\Leftarrow$: Since $\mathrm{cl}(G(f|_{[0,t)}))$ is a subcontinuum of $G(f|_{[0,t]})$ that meets $\{0\} \times f(0)$ and $\{t\} \times f(t)$, we have that $\{t\} \times f(t) \subset \mathrm{cl}(G(f|_{[0,t)}))$. So, f is left cohesive at t. □

In addition to being used in the proofs of several of our results, Observation 1 allows a reader not familiar with the left (right) cohesive fiber concept to think, instead, of left (right) C-set fibers whenever the graph of a continuum-valued function is hereditarily unicoherent and contains no two-sided triod. This is always the case in the proofs of our results.

4. Necessary conditions for chainability of inverse limits

In this section, we provide necessary conditions, for chainability of an inverse limit $X = \varprojlim\{[0,1], f_i\}$ with surjective interval-valued bonding functions, that only involve properties of the bonding functins.

Lemma 1. *If X is a triod, $f\colon X \to Y$ is a continuum-valued function, and $G(f)$ is unicoherent, then $G(f)$ is a triod.*

Proof. Since f is continuum-valued, $c_1\colon G(f) \to X$ is a monotone mapping. By [6, Table IV], X is unicoherent. So, we let $X = J_1 \cup J_2 \cup J_3$ as in the definition of a unicoherent triod. Let $p \in J_1 \cap J_2 \cap J_3$, $x \in J_1 \setminus (J_2 \cup J_3)$,

$y \in J_2 \setminus (J_1 \cup J_3)$, and $z \in J_3 \setminus (J_1 \cup J_2)$. For $i = 1, 2, 3$, let $L_i = G(f|_{J_i})$. Since f is continuum-valued, L_i is a continuum for $i = 1, 2, 3$. Clearly, $G(f) = L_1 \cup L_2 \cup L_3$. To see that $G(f)$ is a triod, let $p' \in f(p)$, $x' \in f(x)$, $y' \in f(y)$, and $z' \in f(z)$. It is immediate that $(p, p') \in L_1 \cap L_2 \cap L_3$, $(x, x') \in L_1 \setminus (L_2 \cup L_3)$, $(y, y') \in L_2 \setminus (L_1 \cup L_3)$, and $(z, z') \in L_3 \setminus (L_1 \cup L_2)$. Hence, $G(f)$ is a triod. □

A set-valued function $f\colon [0,1] \to [0,1]$ has a *flat spot at* p if $p \in [0,1]$ and there exists a nondegenerate interval $I \subset [0,1]$ such that $I \times \{p\} \subset G(f)$. Let $X = \varprojlim\{[0,1], f_i\}$ with surjective, set-valued bonding functions. For $1 < j < i$, *a flat spot at* x_i *for* f_i *composes to a nondegenerate value of* f_j in the composition $f_j \circ f_{j+1} \circ \ldots \circ f_i$ if $f_j(x_i)$ is nondegenerate for $j = i - 1$, or if there exists a point x_{j+1} in $f_{j+1} \circ \ldots \circ f_{i-1}(x_i)$ such that $f_j(x_{j+1})$ is nondegenerate for $j < i - 1$.

Lemma 2. *Let* $X = \varprojlim\{[0,1], f_i\}$ *where each* f_i *is a surjective, interval-valued function. Suppose, for each* $i \geq 1$, $\dim(G(f_i)) = 1$, *and no flat spot of* f_i *composes to a non-degenerate value of* f_j *for* $1 \leq j < i$. *If, for some* $n \geq 2$, G_1^{n+1} *is chainable, then for* $1 \leq i \leq n$, G_i^{n+1} *is chainable and hereditarily decomposable.*

Proof. Let $n \geq 2$, and suppose that G_1^{n+1} is chainable. Let $1 \leq i \leq n$. It follows from [8, Corollary 4] that G_i^{n+1} is a λ-dendroid. Let $G_1^{i-1} \xleftarrow{F} G_i^{n+1}$ be the set-valued function defined by $(x_1, \ldots, x_{i-1}) \in F(x_i, \ldots, x_{n+1})$ if and only if $(x_1, \ldots, x_i, \ldots, x_{n+1}) \in G_1^{n+1}$. This function is defined in [7, 4$^{\text{th}}$ paragraph, page 218], and it is shown to be continuum-valued in Theorem 5 of [7]. Since $G(F) = G_1^{n+1}$ is chainable, it is hereditarily unicoherent and atriodic.

Suppose G_i^{n+1} contains a triod T. Then $G(F|_T) \subset G(F)$, and hence, $G(F|_T)$ is unicoherent. By Lemma 1, $G(F|_T)$ is a triod in $G(F)$, which is a contradiction. So, G_i^{n+1} is an atriodic λ-dendroid, and by [1, Theorem 11], G_i^{n+1} is chainable. The proof is complete. □

The definitions below provide terminology for concepts that are essential in proofs of results in this section.

Definition 6. Let $f\colon [0,1] \to [0,1]$ be a surjective interval-valued function. We say f *has a down* (*up*) *fold at the point* $(s,t) \in G(f)$ if there exist

$u < v$ with $s \in [u,v]$, and interval-valued functions $f'\colon [u,s] \to [0,t]$ and $f''\colon [s,v] \to [0,t]$ ($f'\colon [u,s] \to [t,1]$ and $f''\colon [s,v] \to [t,1]$) such that

(1) $G(f') \cup G(f'') \subset G(f)$,

(2) $(s,t) \in G(f') \cap G(f'')$, and

(3) $G(f') \setminus G(f'') \neq \emptyset \neq G(f'') \setminus G(f')$.

We refer to $G(f') \cup G(f'')$ as a *fold of f at* (s,t).

We note that if $s \in \{u,v\}$, one of f' or f'' will have $\{s\}$ as its domain, in which case, by (3), its value at s is nondegenerate. In this case, we call $G(f') \cup G(f'')$ a *one-sided fold* at (s,t). If $u < s < v$, we call $G(f') \cup G(f'')$ a *two-sided fold* at (s,t). The graph of f_0 in Example 1 of section 6 has a one-sided down fold at $(\frac{1}{2},1)$ and a one-sided up fold at $(\frac{1}{2},0)$.

Definition 7. Let $f\colon [0,1] \to [0,1]$ be a surjective, interval-valued function. Suppose $f(t)$ is a nondegenerate interval for some $t \in [0,1]$. If f is left (right) cohesive at t, and there exists $r < t$ ($r > t$) such that $f([r,t]) = f(t)$, we say that $G(f)$ has *a restricted left (right) cohesive subgraph at* $\{t\} \times f(t)$. Specifically, we say $G(f|_{[r,t]})$ *is a restricted left (right) cohesive subgraph of* $G(f)$ *at* $\{t\} \times f(t)$.

The graph of the function ℓ_2, in Figure 3 of section 6, has a restricted right cohesive subgraph at $\frac{1}{2}$; specifically note that $\ell_2(\frac{1}{2}) = \ell_2([\frac{1}{2},1])$.

Let $\{[0,1], f_i\}$ be an inverse sequence with surjective, interval-valued functions.

Definition 8. For $1 \leq j+1 < n$ and $(t_{j+1},\ldots,t_n) \in G'(f_{j+1},\ldots,f_{n-1})$, a *side-to-side sequence from t_n to t_{j+1}* is a sequence of nondegenerate intervals $I_{j+1},\ldots,I_n$ such that

(i) for $j < i \leq n$, either $t_i \in I_i \subset [0,t_i]$ or $t_i \in I_i \subset [t_i,1]$, and

(ii) for $j < i < n$, $f_i(I_{i+1}) = I_i$.

The points $t_{j+1},\ldots,t_n$ are called *side points* of the intervals $I_{j+1},\ldots,I_n$.

Definition 9. For $1 \leq j+1 \leq n$, a *fold* Λ *of f_n at* (t_n,t_{n+1}) *composes to a side of* $t_{j+1} \in [0,1]$ *where f_j is not cohesive at* t_{j+1} provided that either

(i) $j+1=n$, and f_{n-1} is not left (right) cohesive at t_n when Λ is a down (up) fold, or

(ii) $j+1<n$, and there is a side-to-side sequence from t_n to t_{j+1}, where $c_2(\Lambda)=I_n$, and f_j is not cohesive on the I_{j+1} side of t_{j+1}.

Examples of the notions in Definitions 8 and 9 can be found in the third paragraph of item 2 in Example 1 of section 6.

Definition 10. For $1 \leq j+1 \leq n$, *a restricted left (right) cohesive subgraph* $G(f_n|_{[r_{n+1},t_{n+1}]})$ *of* $G(f_n)$ *at* $f_n(t_{n+1}) \times \{t_{n+1}\}$ *composes to an interval* $I_{j+1} = [r_{j+1}, t_{j+1}]$ *where* f_j *is not cohesive on the* I_{j+1} *side of each of* r_{j+1} *and* t_{j+1} provided that for $I_n = f_n(t_{n+1}) = [r_n, t_n]$ either

(i) $j+1=n$, and f_{n-1} is not cohesive on the I_n side of each of r_n and t_n, or

(ii) $j+1<n$, there exist $(r_{j+1},\ldots,r_n),(t_{j+1},\ldots,t_n) \in G'(f_{j+1},\ldots,f_{n-1})$, and a sequence of nondegenerate intervals $I_{j+1},\ldots,I_n$ such that, for $j<i<n$, $I_i=[r_i,t_i]=f_i(I_{i+1})$, and f_j is not cohesive on the I_{j+1} side of each of r_{j+1} and t_{j+1}.

It may be helpful to a reader to refer to the discussion related to Examples 1 and 3 in section 6 while reading, respectively, the proofs of Cases 1 and 2 in Theorem 1.

Theorem 1. *Let* $X = \varprojlim\{[0,1], f_i\}$, *where for each* $i \geq 1$, f_i *is a surjective, interval-valued function. Properties* (1), (2), (3), *and* (4) *below are necessary conditions for chainability of* X.

(1) $G(f_i)$ *is chainable for each* $i \geq 1$.

(2) *If* $1 \leq j < n$, *no flat spot of* f_n *composes to a nondegenerate value of* f_j.

(3) *If* $1 \leq j+1 \leq n$, *no fold* Λ *of* f_n *composes to a side of* t_{j+1} *where* f_j *is not cohesive at* t_{j+1}.

(4) *If* $1 \leq j+1 \leq n$, *no restricted cohesive subgraph of* $G(f_n)$ *at* $f_n(t_{n+1}) \times \{t_{n+1}\}$ *composes to an interval* $I_{j+1}=[r_{j+1},t_{j+1}]$ *where* f_j *is not cohesive on the* I_{j+1} *side of each of* r_{j+1} *and* t_{j+1}.

Proof. Assume X is chainable. Property (1) follows from [9, Theorem 3]. It also follows from [9, Theorem 3] that each partial graph G_1^{n+1} is chainable. Property (2) follows from [8, Corollary 3]. So, by Lemma 2, for $1 \leq i \leq n$, G_i^{n+1} is chainable. The proofs that properties (3) and (4) follow from the chainability of X are not as immediate. We consider the proofs separately in Cases 1 and 2 below.

Case 1. Assume (3) is not the case. We construct a triod in G_j^{n+1}, giving us a contradition. Assume there exist $n \geq 2$, and $t_n \in f_n(t_{n+1})$ where f_n has a fold V at (t_n, t_{n+1}) that composes, for some $j+1 \leq n$, to a side of t_{j+1} where f_j is not cohesive at t_{j+1}. In particular, we assume, without loss of generality, that $j+1 < n$, V is an up fold of f_n, and there exists a side-to-side sequence from t_n to t_{j+1}, where f_j is not right cohesive at t_{j+1}. The case for $j+1 = n$ is similar, but easier.

By definition of an up fold, there exist a nondegenerate interval $[u_{n+1}, v_{n+1}]$ with $t_{n+1} \in [u_{n+1}, v_{n+1}]$, and interval-valued functions $[t_n, 1] \xleftarrow{f_n'} [u_{n+1}, t_{n+1}]$ and $[t_n, 1] \xleftarrow{f_n''} [t_{n+1}, v_{n+1}]$. So, $V = G(f_n') \cup G(f_n'')$, and $(t_n, t_{n+1}) \in G(f_n') \cap G(f_n'')$. Let $(t_{j+1}, \dots, t_n)$ and $I_{j+1}, \dots, I_n$ be given as in Definition 8. We also have from Definitions 6, 8, and 9 that $t_n \in c_2(V) = I_n \subset [t_n, 1]$ and $I_{j+1} \subset [t_{j+1}, 1]$. Since f_j is not right cohesive at t_{j+1}, we have that $f_j(t_{j+1}) \times \{t_{j+1}\} \not\subset \mathrm{cl}(G(f_j|_{(t_{j+1},1]}))$. Let $K = \mathrm{cl}(G(f_j|_{I_{j+1} \setminus \{t_{j+1}\}}))$. Let $w_j \in f_j(t_{j+1})$ with $(w_j, t_{j+1}) \notin K$, and let $(t_j, t_{j+1}) \in K$. So, K is a subcontinuum of $G(f_j)$, and $c_1(K) = I_{j+1}$. Finally, let $[0,1] \xleftarrow{\hat{f}_j} I_{j+1}$ be the interval-valued function whose graph is K.

By (3) of Definition 6, we may pick points $(a_n, a_{n+1}) \in G(f_n') \setminus G(f_n'')$, and $(b_n, b_{n+1}) \in G(f_n'') \setminus G(f_n')$ with $a_{n+1} \neq b_{n+1}$. Additionally, we note that these two points can be chosen so that $a_n \neq t_n$ and $b_n \neq t_n$. To see this, suppose for each $s \in [u_{n+1}, t_{n+1}]$, $f_n'(s) = \{t_n\}$. Then $u_{n+1} \neq t_{n+1}$, for otherwise $G(f_n') = \{(t_n, t_{n+1})\}$, violating property (3) of Definition 6. So, $\{t_n\} \times [u_{n+1}, t_{n+1}]$ is a flat spot for f_n', and $f_j(t_{j+1})$ is nondegenerate since (w_j, t_{j+1}) and (t_j, t_{j+1}) are two points of $f_j(t_{j+1})$. This violates property (2) of this theorem, which was established in the first paragraph of the proof. A similar argument applies to the interval $[t_{n+1}, v_{n+1}]$. Hence, we can pick (a_n, a_{n+1}) and (b_n, b_{n+1}) as claimed. We assume, without loss of generality, that $c_2(G(f_n')) = I_n$, and $c_2(G(f_n'')) = I_n' \subset I_n$. For each $j+1 \leq i < n$, let $I_i' = f_i(I_{i+1}')$.

For $j \leq i \leq n-2$, let $\hat{f}_{i+1} = f_{i+1}|_{I_{i+2}}$, and let $\tilde{f}_{i+1} = f_{i+1}|_{I_{i+2}'}$. Let

$J_1 = G'(\hat{f}_j, \hat{f}_{j+1}, \ldots, \hat{f}_{n-1}, f'_n)$, $J_2 = G'(\hat{f}_j|_{I'_{j+1}}, \tilde{f}_{j+1}, \ldots, \tilde{f}_{n-1}, f''_n)$, and $J_3 = f_j(t_{j+1}) \times (t_{j+1}, \ldots, t_n, t_{n+1})$. For $i = 1, 2, 3$, $J_i \subset G_j^{n+1}$, and J_i is a continuum since all functions involved in the defintions are interval-valued functions.

We show that $J_1 \cup J_2 \cup J_3$ is a triod in G_j^{n+1}. Recalling the points $(t_j, t_{j+1}) \in K$, $(t_n, t_{n+1}) \in G(f_n)$, and $(t_{j+1}, \ldots, t_n)$ given in the definition of a side-to-side sequence, let $p = (t_j, t_{j+1}, \ldots, t_n, t_{n+1})$. We note that $p \in J_1 \cap J_2 \cap J_3$. Recalling the points $(a_n, a_{n+1}) \in G(f'_n)$ and $(b_n, b_{n+1}) \in G(f''_n)$, we pick points $(a_j, \ldots, a_n) \in G'(\hat{f}_j, \hat{f}_{j+1}, \ldots, \hat{f}_{n-1})$ and $(b_j, \ldots, b_n) \in G'(\hat{f}_j|_{I'_{j+1}}, \tilde{f}_{j+1}, \ldots, \tilde{f}_{n-1})$. Let $a = (a_j, \ldots, a_n, a_{n+1})$, $b = (b_j, \ldots, b_n, b_{n+1})$, and $w = (w_j, t_{j+1}, \ldots, t_n, t_{n+1})$. It is easy to see from the choices of points and defintions that $a \in J_1 \setminus (J_2 \cup J_3)$, $b \in J_2 \setminus (J_1 \cup J_3)$, and $w \in J_3 \setminus (J_1 \cup J_2)$. It follows that $J_1 \cup J_2 \cup J_3$ is a triod in G_j^{n+1}. This is a contradiction since, as noted in the first paragraph of this proof, G_j^{n+1} is chainable.

Case 2. Assume (4) is not the case. We construct a triod in G_j^{n+1}. Assume there exist $n \geq 2$, and a nondegenerate interval $I_{n+1} = [r_{n+1}, t_{n+1}]$ such that $G(f_n|_{I_{n+1}})$ is a restricted left cohesive subgraph of $G(f_n)$ that composes to an interval $I_{j+1} = [r_{j+1}, t_{j+1}]$ with $j + 1 \leq n$, where f_j is not cohesive on the I_{j+1} side of each of r_{j+1} and t_{j+1}. We assume $j + 1 < n$, and $(r_{j+1}, \ldots, r_n)$, $(t_{j+1}, \ldots, t_n)$, and the sequence of intervals $I_{j+1}, \ldots, I_n$ are given as in Definition 10.

Let $[0, 1] \xleftarrow{\hat{f}_j} I_{j+1}$ be the continuum-valued function whose graph is $\mathrm{cl}(G(f_j|_{(r_{j+1}, t_{j+1})}))$. By assumption in this case, $f_j(r_{j+1}) \times \{r_{j+1}\} \not\subset G(\hat{f}_j)$ and $f_j(t_{j+1}) \times \{t_{j+1}\} \not\subset G(\hat{f}_j)$. So, we pick points $v_j \in f_j(r_{j+1})$ and $w_j \in f_j(t_{j+1})$ so that neither (v_j, r_{j+1}) nor (w_j, t_{j+1}) is in $G(\hat{f}_j)$. Also, let (r_j, r_{j+1}) and (t_j, t_{j+1}) be points of $G(\hat{f}_j)$. For $j + 1 \leq i \leq n$, let $\hat{f}_i = f_i|I_{i+1}$. Recall, from Definitions 7 and 10, that $f_n(t_{n+1}) = I_n = f_n(I_{n+1})$.

Define $J_1 = G'(\hat{f}_j, \ldots, \hat{f}_n)$, and let $(x_j \ldots, x_n, r_{n+1}) \in J_1$.

Given $(t_{j+1}, \ldots, t_n, t_{n+1})$ and $(r_{j+1}, \ldots, r_n, t_{n+1})$ in $G'(\hat{f}_{j+1}, \ldots, \hat{f}_n)$, define

$$J_2 = (f_j(r_{j+1}) \times (r_{j+1}, \ldots, r_n, t_{n+1})) \cup (G'(\hat{f}_j, \ldots, \hat{f}_{n-1}) \times \{t_{n+1}\}), \text{ and}$$

$$J_3 = (f_j(t_{j+1}) \times (t_{j+1}, \ldots, t_n, t_{n+1})) \cup (G'(\hat{f}_j, \ldots, \hat{f}_{n-1}) \times \{t_{n+1}\}).$$

We observe that $J_1 \cup J_2 \cup J_3$ is a triod in G_j^{n+1}. We note that, for $i = 1, 2, 3$, J_i is a subcontinuum of G_j^{n+1}. Since $(v_j, r_{j+1}) \notin G(\hat{f}_j)$ and $r_{j+1} \neq t_{j+1}$, we have that $(v_j, r_{j+1}, \ldots, r_n, t_{n+1}) \in J_2 \setminus (J_1 \cup J_3)$. Similarly, since $(w_j, t_{j+1}) \notin G(\hat{f}_j)$ and $r_{j+1} \neq t_{j+1}$, we have that $(w_j, t_{j+1}, \ldots, t_n, t_{n+1}) \in J_3 \setminus (J_1 \cup J_2)$.

Also, since $r_{n+1} \neq t_{n+1}$, we have that $(x_j, \ldots, x_n, r_{n+1}) \in J_1 \setminus (J_2 \cup J_3)$. The point $(t_j, t_{j+1}, \ldots, t_n, t_{n+1})$ is in $J_1 \cap J_2 \cap J_3$. Hence, $J_1 \cup J_2 \cup J_3$ is a triod in G_j^{n+1}, which is a contradiction. $\square$

Remark 1. It is useful to note that, from the proof of Theorem 1, the existence of folds or restricted cohesive subgraphs that compose to sides of values where a bonding function is not cohesive produces triods in the partial graphs. That is, if either (3) or (4) does not hold for some j and n where $1 \leq j+1 \leq n$, then G_j^{n+1} contains a triod.

Question 1. *Let* $X = \varprojlim\{[0,1], f_i\}$*, where for each* $i \geq 1$*,* f_i *is a surjective, interval-valued function. Do properties* (1)*,* (2)*,* (3)*, and* (4) *of Theorem 1 characterize chainability of* X*?*

Remark 2. By Theorem 1, one only needs to establish sufficiency of properties (1) through (4) to have an affirmative answer to Question 1. By Corollaries 3 and 4 in [8], properties (1) and (2) give us that X is tree-like, and each partial graph G_j^n is a λ-dendroid. By [2, Corollary 4.3], if each G_1^n is chainable, then X is chainable. By [1, Theorem 11], each atriodic λ-dendroid is chainable. Hence, an affirmative answer to Question 1 can be established by showing that properties (3) and (4), together with having each G_j^n be a λ-dendriod, are sufficient for atriodicity of the partial graphs G_1^n.

5. A characterization of chainability

Theorem 5 of this section gives a characterization of chainable inverse limits on $[0,1]$ with interval-valued functions that involves properties of the bonding functions f_i and the induced functions F_n defined in section 2. Although such a characterization is not as desirable as one that only involves properties of the bonding functions, Corollary 1 of this section gives sufficient conditions on the bonding functions for a chainable inverse limit. The examples in section 6 illustrate how Corollary 1 can be used to determine chainability. Also, Theorem 5 characterizes how triods arise in the partial graphs.

5.1. Observations and theorems related to a finite inverse sequence $Y \xleftarrow{g} [0,1] \xleftarrow{f} [0,1]$

Given a continuum Y, let $Y \xleftarrow{g} [0,1] \xleftarrow{f} [0,1]$ be a finite inverse sequence, where each of f and g is a surjective continuum-valued function. The *induced function* $G(g) \stackrel{F}{\longleftarrow} [0,1]$ is defined by $F(t) = G(g|_{f(t)})$ for $t \in [0,1]$. We observe that $G(F) = G'(g,f)$. Furthermore, since both f and g are continuum-valued, it follows that for each $t \in [0,1]$, $G(g|_{f(t)})$ is a continuum. Hence, F is continuum-valued. As discussed in section 2, notation for graphs of functions related to this inverse sequence will be relative to the order in the inverse sequence.

For all results in this subsection, we let $Y \stackrel{g}{\longleftarrow} [0,1] \stackrel{f}{\longleftarrow} [0,1]$ be a finite inverse sequence, where each of f and g is a surjective continuum-valued function, each of $G(f)$ and $G(g)$ is chainable, and $G'(g,f)$ is hereditarily unicoherent. Since $G(F) = G'(g,f) \subset Y \times [0,1] \times [0,1]$, we let, for $1 \leq i < j \leq 3$, π_i and $\pi_{i,j}$ denote, respectively, the projection mappings from $G'(g,f)$ onto the i^{th} coordinate, and onto the i^{th} and j^{th} coordinates.

Observation 2 can be useful for determining left or right cohesiveness of induced functions in specific examples of inverse sequences. We illustrate this in the examples in section 6. Observations 3, 4, and 5 are useful in the proofs of Theorems 2 and 3.

Observation 2. *Let $t \in [0,1]$. If both $f(t)$ and $g(f(t))$ are degenerate, then F is cohesive at t. If f is fully left (right) cohesive at t, then F is fully left (right) cohesive at t.*

Proof. If both $f(t)$ and $g(f(t))$ are degenerate, then $F(t) = (g(f(t)), f(t))$ is degenerate, and clearly F is cohesive at t. We assume f is fully left cohesive at t. Let $(y,w,t) \in F(t) \times \{t\}$. So, $w \in f(t)$ and $y \in g(w)$. By assumption, there exists an increasing sequence of points $\{t_i\}_{i \geq 1}$ converging to t, where, for each $i \geq 1$, $w \in f(t_i)$. So, for each $i \geq 1$, $(y,w) \in F(t_i)$, and $\{(y,w,t_i)\}_{i \geq 1}$ converges to (y,w,t), giving us that F is fully left cohesive at t. □

Observation 3. *Let K be a subcontinuum of $G'(g,f)$.*

(*a*) *If $\pi_3(K) = [r,t]$, then* $\mathrm{cl}(G(f|_{(r,t)})) \subset \pi_{2,3}(K)$.

(*b*) *If $\pi_2(K) = [u,v]$, then* $\mathrm{cl}(G(g|_{(u,v)})) \subset \pi_{1,2}(K)$.

Proof. (a) By hypothesis, $\pi_{2,3}(K) \subset G(f|_{[r,t]})$. Suppose, for some $r < s < t$, $f(s) \times \{s\} \not\subset \pi_{2,3}(K)$. Then clearly $\pi_{2,3}(K) \cup (f(s) \times \{s\})$ is a two-sided triod in $G(f)$, which contradicts the chainability of $G(f)$. So, $G(f|_{(r,t)}) \subset \pi_{2,3}(K)$. Since $\pi_{2,3}(K)$ is a continuum, we have that $\mathrm{cl}(G(f|_{(r,t)})) \subset \pi_{2,3}(K)$.

(b) The proof is similar to the proof of (a). □

Observation 4. *If $t \in [0,1]$, and $f(t) = [u,v]$ is nondegenerate, then*

(*a*) *for $t > 0$, $\mathrm{cl}(G(f|_{[0,t)}))$ contains either (u,t) or (v,t), and for $t < 1$, $\mathrm{cl}(G(f|_{(t,1]}))$ contains either (u,t) or (v,t), and*

(*b*) *if $\mathrm{cl}(G(f|_{[0,t)}))$ does not contain (w,t) for $w \in \{u,v\}$, then $(w,t) \in \mathrm{cl}(G(f|_{(t,1]}))$. An analogous statement holds for $\mathrm{cl}(G(f|_{(t,1]}))$.*

Proof. (a) Suppose $\mathrm{cl}(G(f|_{[0,t)}))$ contains neither (u,t) nor (v,t). Then it is easy to see that $\mathrm{cl}(G(f|_{[0,t)})) \cup ([u,v] \times \{t\})$ is a one-sided triod in $G(f)$, a contradiction.

(b) Suppose (v,t) is not in $\mathrm{cl}(G(f|_{[0,t)})) \cup \mathrm{cl}(G(f|_{(t,1]}))$. Then $\mathrm{cl}(G(f|_{[0,t)})) \cup \mathrm{cl}(G(f|_{(t,1]})) \cup ([u,v] \times \{t\})$ is a triod in $G(f)$, a contradiction. □

Observation 5. *Let K be a subcontinuum of $G'(g,f)$.*

(*a*) *If $0 < w < 1$, and there exists y such that $(y,w) \notin \pi_{1,2}(K)$, then $w \notin \mathrm{int}(\pi_2(K))$.*

(*b*) *If $\pi_3(K) = [s,t]$ is a nondegenerate interval, and there exists y such that $(y,w,t) \in G'(g,f) \setminus K$, then there exists $r \neq t$ and a subcontinuum K_r of K such that $\pi_3(K_r) = [r,t]$ and $(y,w) \notin \pi_{1,2}(K_r)$.*

Proof. (a) Suppose $0 < w < 1$, and $w \in \mathrm{int}(\pi_2(K))$. Then $\pi_{1,2}(K)$ is a continuum in $G(g)$ that meets both sides of $g(w) \times \{w\}$ in $G(g)$. By Observation 3(b), $(y,w) \in \pi_{1,2}(K)$, which is a contradiction.

(b) If $w \notin \pi_2(K)$, then $(y,w) \notin \pi_{1,2}(K)$; so, we may choose $r = s$ and $K_r = K$. We assume $w \in \pi_2(K)$. Assume, without loss of generality, that $s < t$, and no such $r < t$ exists. Let $\{t_i\}$ be an increasing sequence in $[0,1]$ converging to t with $t_1 > s$. For each $i \geq 1$, let $K_i = K \cap \pi_3^{-1}([t_i,t])$. By assumption, for each $i \geq 1$, there exists a point $(y,w,r_i) \in K_i$. It follows, from hereditary unicoherence of $G'(g,f)$, that each K_i is a subcontinuum of K. We note also that $\{K_i\}$ is a nested decreasing sequence whose intersection lies in $\pi_3^{-1}(t)$. So, $\{r_i\}$ converges to t, and $\{(y,w,r_i)\}$ converges to (y,w,t), putting $(y,w,t) \in K$, a contradiction. □

Theorem 2. *Let $t \in [0,1]$ where $f(t) = \{u\}$ is degenerate. Then*

(1) *$G'(g,f) = G(F)$ contains no one-sided triod at t, and*

(2) *if $G'(g,f) = G(F)$ contains a two-sided triod at t, then f has either a two-sided down fold or a two-sided up fold at (u,t) and g is, respectively, either not left or not right cohesive at u.*

Proof. We note that if p is a flat spot for f, then $g(p)$ is a singleton, for otherwise, $G'(g,f)$ is not hereditarily unicoherent.

(1) Suppose $G'(g,f) = G(F)$ contains a one-sided triod at t. By definition of a one-sided triod, we may assume, without loss of generality, that there exist a nondegenerate interval $[s,t]$ with $s < t$, two nondegenerate subcontinua A and B of $\pi_3^{-1}(t) = g(u) \times \{u\} \times \{t\}$, and a subcontinuum K of $G'(g,f)$ such that $\pi_3(K) = [s,t]$, and $A \cup B \cup K$ is a triod in the hereditarily unicoherent continuum $G'(g,f)$. Let $(a,u,t) \in A \setminus (B \cup K)$, and $(b,u,t) \in B \setminus (A \cup K)$. By Observation 5(b), we can choose $s \leq r < t$ and a subcontinuum K_r of K so that (a,u) and (b,u) are not in $\pi_{1,2}(K_r)$. It is clear that $K_r \cup A \cup B$ is also a one-sided triod in $G'(g,f)$. We note that $\pi_2(K_r)$ is nondegenerate; for otherwise, $\{u\} \times [r,t]$ is a flat spot of f where $g(u)$ is nondegenerate, contradicting the first paragraph of the proof. So, $\pi_2(K_r) \neq \{u\}$. From this, it is easy to see that $\pi_{1,2}(A) \cup \pi_{1,2}(B) \cup \pi_{1,2}(K_r)$ is a triod in $G(g)$, which contradicts the chainability of $G(g)$.

(2) Suppose $G'(g,f) = G(F)$ contains a two-sided triod at t. By definition, there exist a nondegenerate interval $[r,s]$ with $r < t < s$, and subcontinua K_r and K_s of $G'(g,f)$ such that $\pi_3(K_r) = [r,t]$ and $\pi_3(K_s) = [t,s]$, and $K_r \cup K_s \cup (F(t) \times \{t\})$ is a triod. So, there is a point (y,u,t) in $F(t) \times \{t\}$ that is not in $K_r \cup K_s$. Since (y,u,t) is not in $K_r \cup K_s$, we apply Observation 5(b), as we did in the proof of (1), to get r', s', and subcontinua $K_{r'}$ and $K_{s'}$ of, respectively, K_r and K_s such that $\pi_3(K_{r'}) = [r',t]$, $\pi_3(K_{s'}) = [t,s']$, and $(y,u) \notin \pi_{1,2}(K_{r'} \cup K_{s'})$. Clearly, $K_{r'} \cup K_{s'} \cup (F(t) \times \{t\})$ is also a two-sided triod. For notational convenience, we let $K_1 = K_{r'}$, $K_2 = K_{s'}$, and $T = K_1 \cup K_2 \cup (F(t) \times \{t\})$. Hereafter, we consider the two-sided triod T.

As in the proof of (1), we have both $\pi_2(K_1)$ and $\pi_2(K_2)$ are nondegenerate intervals. If $\pi_2(K_1 \cup K_2)$ is not a subset of either $[0,u]$ or $[u,1]$, then $u \in \mathrm{int}(\pi_2(K_1 \cup K_2))$, giving us a contradiction to Observation 5(a) since $(y,u) \notin \pi_{1,2}(K_1 \cup K_2)$. So, we suppose, without loss of generality, that $\pi_2(K_1 \cup K_2)$ is a nondegenerate interval in $[0,u]$. Let $(x,u,t) \in K_1 \cap K_2 \cap (F(t) \times \{t\})$. Since $\pi_{2,3}(K_1)$ and $\pi_{2,3}(K_2)$ are subcontinua of $G(f)$ that may be thought of as the graphs of continuum-valued functions, we note that $\pi_{2,3}(K_1) \cup \pi_{2,3}(K_2)$ is a two-sided down fold of f at (u,t). Also, since (y,u) is not in $\pi_{1,2}(K_1)$, and $\pi_{1,2}(K_1)$ is a continuum in $G(g|_{[0,u]})$ that meets both $g(u) \times \{u\}$ and $G(g|_{[0,u)})$, it follows that $g(u) \times \{u\}$ is not a left side C-set in $G(g)$. By Observation 1, g is not left cohesive at u. $\square$

Theorem 3. *Let $t \in [0,1]$ where $f(t) = [u,v]$ is a nondegenerate interval with $u < v$. If $G'(g,f) = G(F)$ contains either a one-sided or two-sided triod at t, then either*

(1) *f has either an up fold at (u,t) or a down fold at (v,t), and g is, respectively, either not right cohesive at u or not left cohesive at v, or*

(2) *$G(f)$ has a restricted left (right) cohesive subgraph at $f(t) \times \{t\}$, and g is both not right cohesive at u and not left cohesive at v.*

Proof. **Case 1.** Suppose $G(F)$ contains a two-sided triod at t. Let $T = K_r \cup K_s \cup (F(t) \times \{t\})$ be a two-sided triod, where $[r,s]$, K, K_r, and K_s are defined analogously as in the proof of Theorem 2(2). Let $(y,w,t) \in (F(t) \times \{t\}) \setminus (K_r \cup K_s)$. We also assume $[r',s'] \subset [r,s]$, and $K_{r'}$, and $K_{s'}$ have been modified using Observation 5(b), as in the proof of Theorem 2(2), so that $(y,w) \notin \pi_{1,2}(K_{r'} \cup K_{s'})$. Lastly, we assume, without loss of generality, that $r' = r$ and $s' = s$.

Let $(q,p,t) \in (F(t) \times \{t\}) \cap K_r \cap K_s$. By Observation 4, $\mathrm{cl}(G(f|_{(r,t)})) \cup \mathrm{cl}(G(f|_{(t,s)}))$ contains both (u,t) and (v,t). By Observation 3(a), $\pi_{2,3}(K_r) \cup \pi_{2,3}(K_s)$ contains both (u,t) and (v,t). Also, $(p,t) \in \pi_{2,3}(K_r) \cap \pi_{2,3}(K_s) \cap ([u,v] \times \{t\})$; so, we have that $[u,v] \times \{t\} \subset \pi_{2,3}(K_r \cup K_s)$. Since $(y,w) \notin \pi_{1,2}(K_r \cup K_s)$ and $u \leq w \leq v$, it follows, from Observation 5(a), that $w \in \{u,v\}$. We suppose, without loss of generality, that $w = u$. So, $(y,u) \notin \pi_{1,2}(K_r \cup K_s)$.

We assume, without loss of generality, that $(u,t) \in \pi_{2,3}(K_r)$. It follows from Observation 5(a) that $u \notin \mathrm{int}(\pi_2(K_r \cup K_s))$. From the previous paragraph, both u and v are in $\pi_2(K_r \cup K_s)$. Since $u < v$, we have that, $\pi_2(K_r \cup K_s) \subset [u,1]$. Let $K_r' = \pi_{2,3}(K_r) \cup ([u,v] \times \{t\})$, and $K_s' = \pi_{2,3}(K_s) \cup ([u,v] \times \{t\})$. So, we have that $K_r' \cup K_s'$ is a two-sided up fold of f at (u,t). Recall that $(y,u) \notin \pi_{1,2}(K_r)$, and $\pi_{1,2}(K_s)$ is a subcontinuum of $G(g|_{[u,1]})$ that meets both $g(u) \times \{u\}$ and $G(g|_{(u,1]})$. From this, we note that $g(u) \times \{u\}$ is not a right side C-set in $G(g)$. By Observation 1, g is not right cohesive at u, giving us that (1) of the conclusion holds.

Case 2. Suppose $G(F)$ contains a one-sided triod at t. Let $T = A \cup B \cup K$ be a triod, where A and B are subcontinua of $F(t) \times \{t\}$, and K is a subcontinuum of $G(F)$ such that $\pi_3(K) = [s,t]$ with $s < t$. By Observation 3(a), $\mathrm{cl}(G(f|_{(s,t)}) \subset \pi_{2,3}(K)$.

By Observation 4(a), we may assume, without loss of generality, that $(u,t) \in \mathrm{cl}(G(f|_{[0,t)}))$. From the previous paragraph, $(u,t) \in \pi_{2,3}(K)$. Let $(x,a,t) \in A \setminus (B \cup K)$ and $(y,b,t) \in B \setminus (A \cup K)$. Let $(z,w,t) \in A \cap B \cap K$ be chosen so that w is the largest point of $\pi_2(A \cap B \cap K)$ in the natural order on $[0,1]$. The choices of these three points put $a, b, w \in f(t) = [u,v]$. Also, $[u,w] \subset \pi_2(K)$. As we have done several times, we modify K by applying Observation 5(b) to get a subcontinuum K_r so that (x,a) and (y,b) are not in $\pi_{1,2}(K_r)$. We may assume, without loss of generality, that $K_r = K$ and $r = s$ from the first paragraph of this case.

By Observation 5(a), neither a nor b is in the interior of $\pi_2(K)$. So, in particular, neither a nor b is in the open segment in $[0,1]$ from u to w. We assume, without loss of generality, that $a \leq b$.

Suppose $u < a < b$. From the previous two paragraphs, we have that $[u,w] \subset \pi_2(K)$ and $[w,b] \subset \pi_2(B)$. Hence, we have $a \in \mathrm{int}(\pi_2(K \cup B))$ with $(x,a) \notin \pi_{1,2}(K \cup B)$, contradicting Observation 5(a). So, we must have that either $a = b$, or $a = u$ and b is not in $[u,w)$ since b is not in the interior of $\pi_2(K)$.

(a) Assume $a = u$ and b is not in $[u,w)$. From above, we have that $(x,a) = (x,u) \notin \pi_{1,2}(K)$, which contains $\mathrm{cl}(G(g|_{(u,w)})$ by Observation 3(b). So, g is not right cohesive at u. Also, by Observation 5(a), $u \notin \mathrm{int}(\pi_2(K))$, so, $f([s,t]) \subset [u,1]$. If f is not left cohesive at t, then $G(f|_{[s,t]}) \cup (f(t) \times \{t\})$ is a one-sided up fold of f at (u,t), and we have the desired conclusion.

So, we assume that f is left cohesive at t. From this assumption and Observation 3(b), we have that $[u,v] \times \{t\} \subset \mathrm{cl}(G(f|_{(s,t)})) \subset \pi_{2,3}(K)$. So, $[u,v] \subset \pi_2(K)$. As observed in the first sentence of the third previous paragraph, $b \notin \mathrm{int}(\pi_2(K))$. Also, by assumption in this case, $b \neq u$. Hence, $b = v$, and $v \notin \mathrm{int}(\pi_2(K))$. So, $f([s,t]) \subset [0,v]$. We now have that $f([s,t]) = [u,v]$. Also, by Observation 1, we observe that g is not left cohesive at v since $\pi_{1,2}(K)$ is a continuum in $G(g)$ such that $\pi_2(K) = [u,v]$, but $(y,b) = (y,v) \notin \pi_{1,2}(K)$. We see that $G(f|_{[s,t]})$ is a restricted left cohesive subgraph of $G(f)$, and (2) of the conclusion is satisfied.

(b) Assume $a = b$. Then $x \neq y$ since $(x,a) \neq (y,b)$.

Suppose $u \neq a$. Then $a \geq w$. Recall that $(z,w,t) \in K \cap A \cap B$. Let $[0,1] \xleftarrow{\hat{f}} \{t\}$ be the function such that $\hat{f}(t) = [w,a]$. Let $A' = G'(g|_{[w,a]}, \hat{f}) \cap A$, and $B' = G'(g|_{[w,a]}, \hat{f}) \cap B$. So, $(z,w) \in \pi_{1,2}(K) \cap \pi_{1,2}(A') \cap \pi_{1,2}(B')$.

Also, $(x,a) \in \pi_{1,2}(A') \setminus (\pi_{1,2}(B') \cup \pi_{1,2}(K))$, and $(y,a) \in \pi_{1,2}(B') \setminus (\pi_{1,2}(A') \cup \pi_{1,2}(K))$. For any point $(p,u,t) \in K$, we have $(p,u) \in \pi_{1,2}(K) \setminus (\pi_{1,2}(A') \cup \pi_{1,2}(B'))$. Hence, $\pi_{1,2}(K) \cup \pi_{1,2}(A') \cup \pi_{1,2}(B')$ is a triod in $G(g)$, which is a contradiction.

Suppose $u = a = b$. Then $\pi_2(K) \neq \{u\}$; for otherwise, since $K \subset G(F)$ and $\pi_3(K) = [s,t]$, $\{u\} \times [s,t]$ would be a flat spot of f at u, where $g(u)$ is nondegenerate, violating the hereditary unicoherence of $G'(g,f)$. Similarly as in the previous paragraph, letting $A' = A \cap (g(u) \times \{u\} \times \{t\})$, and $B' = B \cap (g(u) \times \{u\} \times \{t\})$, we get that $\pi_{1,2}(K) \cup \pi_{1,2}(A') \cup \pi_{1,2}(B')$ is a triod in $G(g)$, which is a contradiction. □

Theorem 4 establishes converse statements to Theorems 2 and 3.

Theorem 4. *Let* $t \in [0,1]$.

(1) *If f has a down (up) fold at (u,t), and g is not (left) right cohesive at u, then $G'(g,f) = G(F)$ contains a triod.*

(2) *If $f(t) = [u,v]$ with $u < v$, and $G(f)$ has a restricted left (right) cohesive subgraph at $f(t) \times \{t\}$, and g is both not right cohesive at u and not left cohesive at v, then $G'(g,f) = G(F)$ contains a triod.*

Proof. (1) Suppose, without loss of generality, that f contains an up fold V at (u,t), and g is not right cohesive at u. So, there exist an interval $[r,s]$ and interval-valued functions $[u,1] \xleftarrow{f'} [r,t]$ and $[u,1] \xleftarrow{f''} [t,s]$ as in Definition 6. So, $V = G(f') \cup G(f'')$, and $(u,t) \in G(f') \cap G(f'')$. Assume, without loss of generality, that $c_2(V) = [u,w]$ with $u < w$. Since g is not right cohesive at u, $g(u) \times \{u\} \not\subset \mathrm{cl}(G(g|_{(u,1]}))$. Let $Y \xleftarrow{\hat{g}} [u,w]$ be the continuum-valued function whose graph is $\mathrm{cl}(G(g|_{(u,w]}))$. Let $x \in \hat{g}(u)$, and let $(y,u) \in G(g) \setminus G(\hat{g})$. Let $T = G'(\hat{g}|_{f'([r,t])}, f') \cup G'(\hat{g}|_{f''([t,s])}, f'') \cup (g(u) \times \{u\} \times \{t\})$. We saw similar constructions of triods in Cases 1 and 2 of Theorem 1. As in the last paragraphs of those two cases, it is easy to see that T is a triod in $G'(g,f) = G(F)$.

(2) Suppose $f(t) = [u,v]$ with $u < v$, $G(f)$ has a restricted left cohesive subgraph at $f(t) \times \{t\}$, and g is both not right cohesive at u and not left cohesive at v. So, by Definition 7, there exists $r < t$ where $f([r,t]) = [u,v]$. Since g is not right cohesive at u and not left cohesive at v, we let $Y \xleftarrow{\hat{g}} [u,v]$ be the continuum-valued function whose graph is $\mathrm{cl}(G(g|_{(u,v)}))$, and we note that there exist points (y_1,u) and (y_2,v) in $G(g) \setminus G(\hat{g})$. Letting

$J_1 = G'(\hat{g}, f|_{[r,t]})$, $J_2 = (G(\hat{g}) \times \{t\}) \cup (g(u) \times \{u\} \times \{t\})$ and $J_3 = (G(\hat{g}) \times \{t\}) \cup (g(v) \times \{v\} \times \{t\})$, we see that $T = J_1 \cup J_2 \cup J_3$ is a triod in $G'(g, f)$. □

5.2. The characterization

Theorem 5. *Let $X = \varprojlim\{[0,1], f_i\}$, where for each $i \geq 1$, f_i is a surjective, interval-valued function. Then X is chainable if and only if properties* (1), (2), (*A*), *and* (*B*) *below hold.*

(1) *$G(f_i)$ is chainable for each $i \geq 1$.*

(2) *If $1 \leq j < n$, no flat spot of f_n composes to a nondegenerate value of f_j.*

(*A*) *If, for some $n \geq 2$, Λ is a down (up) fold of f_n at (t_n, t_{n+1}), then F_{n-1} is left (right) cohesive at t_n.*

(*B*) *If, for some $n \geq 2$, $f_n(t_{n+1}) = [r_n, t_n]$ with $r_n < t_n$, and $G(f_n)$ has a restricted left (right) cohesive subgraph at $f_n(t_{n+1}) \times \{t_{n+1}\}$, then F_{n-1} is either right cohesive at r_n or left cohesive at t_n.*

Proof. As we have seen, properties (1) and (2) are equivalent to having each partial graph G_1^{n+1} be a λ-dendroid. Also, we recall that λ-dendroids are chainable if and only if they are atriodic. Given this, X is chainable if and only if each G_1^{n+1} is atriodic. So, to complete the proof, we only need to see that, assuming (1) and (2), properties (A) and (B) are equivalent to atriodicity of the partial graphs G_1^{n+1}. We prove the necessary implications in Cases 1 and 2 below.

Case 1. (A) and (B) $\Rightarrow$ G_1^{n+1} is atriodic. We prove the contrapositive statement. Suppose, for some $n \geq 1$, G_1^{n+1} contains a triod. If $n = 1$, then $G_1^2 = G(f_1)$ which is chainable. So, we choose the least $n \geq 2$ for which G_1^{n+1} contains a triod. We note that $G_1^{n+1} = G'(F_{n-1}, f_n) = G(F_n)$, and $G(f_n)$ is chainable by (1). Furthermore, $G(F_{n-1}) = G_1^n$ is chainable by the choice of n. The fibers $F_n(t) \times \{t\}$ of $G(F_n) = G_1^{n+1}$ are atriodic since each one is homeomorphic to $F_n(t) \subset G_1^n$. Since the fibers of $G(F_n)$ are atriodic, by [9, Theorem 1], as discussed after Definitions 1 and 2 in section 3, $G(F_n)$ contains either a one-sided or a two-sided triod. Hence, it follows from Theorems 2 and 3 that either (A) or (B) does not hold.

Case 2. G_1^{n+1} is atriodic $\Rightarrow$ (A) and (B). We prove the contrapositive statement. Suppose one of (A) or (B) is not true for some $n \geq 2$. As in

Case 1, we consider $G'(F_{n-1}, f_n)$, and note that the hypotheses of statements (1) and (2) in Theorem 4 are denials of properties (A) and (B). Hence, it follows from Theorem 4 that G_1^{n+1} contains a triod. □

Let $\{[0,1], f_i\}$ be an inverse sequence with surjective interval-valued bonding functions, and let $n \geq 1$. If t is a point of $[0,1]$, by the *n-orbit* of t, we mean the finite sequence of sets $f_1 \circ \ldots \circ f_n(t), \ldots, f_{n-1} \circ f_n(t), f_n(t)$ contained, respectively, in the first through n^{th} factor spaces. If each set in the n-orbit of t is a singleton, it is observed in [5, Lemma 3.2] that $F_n(t)$ is degenerate. For a single set-valued functon $f\colon [0,1] \to [0,1]$, the *full orbit* of t is the sequence of sets $\{f^n(t)\}_{n\geq 1}$, where f^n denotes the composition of f with itself n times.

For Corollary 1, one should recall Definition 4 in section 3.

Corollary 1. *Let $X = \varprojlim\{[0,1], f_i\}$, where for each $i \geq 1$, f_i is a surjective interval-valued function. Suppose* (1) *and* (2) *in Theorem 5. Also, suppose* (*a*) *and* (*b*) *below.*

(*a*) *Whenever f_n has a down (up) fold at (t_n, t_{n+1}) for some $n \geq 2$, either the $(n-1)$-orbit of t_n consists of singletons or f_{n-1} is fully left (right) cohesive at t_n.*

(*b*) *Whenever $f_n(t_{n+1}) = [r_n, t_n]$ with $r_n < t_n$, and $G(f_n)$ has a restricted left (right) cohesive subgraph at $f_n(t_{n+1}) \times \{t_{n+1}\}$ for some $n \geq 2$, we have that one of* (*i*) *or* (*ii*) *holds.*

  (*i*) *Either the $(n-1)$-orbit of r_n consists of singletons, or f_{n-1} is fully right cohesive at r_n.*

  (*ii*) *Either the $(n-1)$-orbit of t_n consists of singletons, or f_{n-1} is fully left cohesive at t_n.*

Then X is chainable.

Proof. If f_n has a down (up) fold at (t_n, t_{n+1}), by (a) and Observation 2, it follows that F_{n-1} is left (right) cohesive at t_n. Similarly, if $f_n(t_{n+1}) = [r_n, t_n]$, and $G(f_n)$ has a restricted left (right) cohesive subgraph at $f_n(t_{n+1}) \times \{t_{n+1}\}$, by (b) and Observation 2, it follows that F_{n-1} is either right cohesive at r_n or left cohesive at t_n. So, (A) and (B) of Theorem 5 are satisfied. Therefore, X is chainable. □

6. Examples

To illustrate the easy use of our results to determine chainability, or the lack thereof, of inverse limits on $[0,1]$ with interval-valued functions, we look at three examples, the first of which is a family of functions considered by Ingram [3, Example 7.1].

Example 1. For $0 \leq a \leq 1$, define $f_a\colon [0,1] \to [0,1]$ as follows. For $0 \leq t < \frac{1}{2}$, let $f_a(t) = 2t$. Let $f_a(\frac{1}{2}) = [a,1]$. For $\frac{1}{2} < t \leq 1$, let $f_a(t) = 2(1-a)t + 2a - 1$. We note that f_1 is a mapping, and, for $a < 1$, f_a has exactly one nondegenerate value at $\frac{1}{2}$. We consider inverse limits $X_a = \varprojlim\{[0,1], f_a\}$ with the single bonding function f_a from this family. Graphs of f_0, $f_{\frac{1}{4}}$, and $f_{\frac{3}{4}}$ are shown in Figure 1. The functions f_0 and $f_{\frac{3}{4}}$ produce chainable inverse limits. The inverse limit produced with $f_{\frac{1}{4}}$ is not chainable.

Ingram showed in [3] that X_a is chainable if and only if $f_a^n(a) \neq \frac{1}{2}$ for $n \geq 0$. He additionally showed that X_0 is indecomposable, and that X_a is an arc for each $\frac{1}{2} < a \leq 1$. He showed X_a is chainable for each a such that $f_a^n(a) \neq \frac{1}{2}$ by proving that each partial graph G_1^{n+1} is an arc. In general, this can be difficult to determine, as Ingram states in the sentence preceeding Question 5.2. We simply wish to observe, in this example, how his chainability characterization for this family follows from our results.

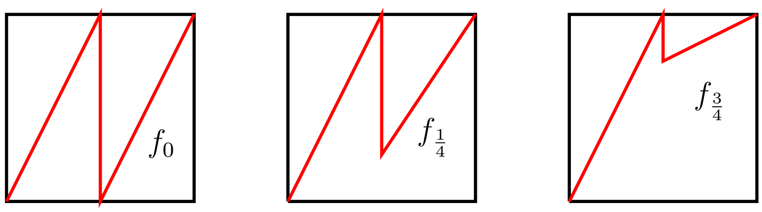

Figure 1. Graphs of f_0, $f_{\frac{1}{4}}$, and $f_{\frac{3}{4}}$.

We begin by observing that, for no $0 \leq a \leq 1$, does $G(f_a)$ contain a restricted cohesive subgraph, so by Theorem 1 and Corollary 1, we only need to consider points in $G(f_a)$ where there is a down or up fold in order to determine chainability or non-chainability of X_a. For each $0 \leq a < 1$, f_a has a down fold at $(\frac{1}{2}, 1)$, has an up fold at $(\frac{1}{2}, a)$, and is both not left and not right cohesive at $\frac{1}{2}$.

1. X_a is chainable if $f_a^n(a) \neq \frac{1}{2}$ for $n \geq 0$. Consider an up fold V of f_a at $(a, \frac{1}{2})$ in the m^{th} factor space for some $m \geq 2$. Since $f_a^n(a) \neq \frac{1}{2}$ for $n \geq 0$, we have that the $(m-1)$-orbit of a from the m^{th} factor space $[0,1]$ is a sequence of singletons. For a down fold Λ at $(1,\frac{1}{2})$ in the m^{th} factor space, we have that the $(m-1)$-orbit of 1 is the constant $\{1\}$ sequence. Thus, by Corollary 1(a), X_a is chainable.

2. X_a is not chainable if, for some $n \geq 0$, $f_a^n(a) = \frac{1}{2}$. If $a = \frac{1}{2}$, let V be the one-sided up fold of $f_{\frac{1}{2}}$ at $(\frac{1}{2},\frac{1}{2})$ where $c_2(V) = [\frac{1}{2},1]$. Since f_a is not right cohesive at $\frac{1}{2}$, it follows from Theorem 1(3) that $X_{\frac{1}{2}}$ is not chainable.

Suppose $a \neq \frac{1}{2}$, and assume that n is the smallest positive interger such that $f_a^n(a) = \frac{1}{2}$. We see, from the definition of f_a, that, for each interval $J = [u,v]$ not containing $\frac{1}{2}$, $f_a([u,v]) = [f_a(u), f_a(v)]$, and the length of $f_a(J)$ is less than or equal to twice the length of J. From this, it should be clear that we can pick $b > a$ where the interval $I_{n+2} = [a,b]$ is small enough so that $I_{n+2-i} = f_a^i(I_{n+2})$ does not contain $\frac{1}{2}$ for each $1 \leq i \leq n-1$. It follows that f_a is an increasing linear mapping on each I_i, and $I_2, I_3, \ldots, I_{n+2}$ is a right side to right side sequence of intervals with side points $\frac{1}{2} = f_a^n(a), \ldots, f_a(a), a$. Finally, we pick the up fold V of f_a at $(a,\frac{1}{2})$ where $c_2(V) = I_{n+2}$. This gives an up fold of f_a that composes to the right side of $\frac{1}{2}$ where f_a is not right cohesive. Thus, by Theorem 1(3), X_a is not chainable.

For clarity, we illustrate this process for f_a when $a = \frac{1}{8}$. Choose $I_4 = [\frac{1}{8},\frac{3}{16}]$. Let V be the up fold of f_a whose first coordinate is in the fifth factor space, and $c_2(V) = I_4$. Then $I_3 = f_a(I_4) = [\frac{1}{4},\frac{3}{8}]$, and $I_2 = f_a(I_3) = [\frac{1}{2},\frac{3}{4}]$. We have a right side to right side sequence of intervals such that the up fold V composes to the right side of $\frac{1}{2}$ in the second factor space where f_a is not right cohesive.

One can find a different proof of the existence of the interval I_{n+2}, discussed in the second paragraph above this one, in [3]. See the last five lines of the paragraph immediately preceeding the definitions of α, β, and γ in Example 7.1 of [3].

In Example 2, we consider a family of functions similar to the family in Example 1, however, the graphs of the functions are not arcs. So, chainability cannot be proven by showing that each partial graph is an arc. We make the family a little easier to analize by having additional restrictions.

Example 2. Let $0 \leq a \leq c < 1$. Define $g_{a,c}\colon [0,1] \to [0,1]$ as follows. Let $G(g_{a,c}|_{[0,\frac{1}{2})})$ be a $\sin\frac{1}{x}$-curve with endpoint $(0,0)$, limit bar $\{\frac{1}{2}\} \times [c,1]$, down folds at a sequence of points $\{(x_i,1)\}$ converging to $(\frac{1}{2},1)$, and up folds at a sequence of points $\{(y_i,c)\}$ converging to $(\frac{1}{2},c)$. Let $g_{a,c}|_{[\frac{1}{2},1]} = f_a$ from

Example 1. Graphs for $g_{\frac{5}{8},\frac{3}{4}}$ and $g_{\frac{1}{4},\frac{1}{2}}$ are shown in Figure 2.

Let $Y_{a,c} = \varprojlim\{[0,1], g_{a,c}\}$ with the single bonding function $g_{a,c}$ from this family. We show that $Y_{a,c}$ is chainable if, for all $n \geq 0$, $g^n_{a,c}(a) \neq \frac{1}{2}$ and $g^n_{a,c}(c) \neq \frac{1}{2}$. As in Example 1, we only need to consider the points where $g_{a,c}$ has either an up fold or a down fold, and show that the second coordinate of each such point does not compose under iterates of $g_{a,c}$ to $\frac{1}{2}$, which is the only value where $g_{a,c}$ is not cohesive. Hence, it will suffice to look at the full orbits of such points.

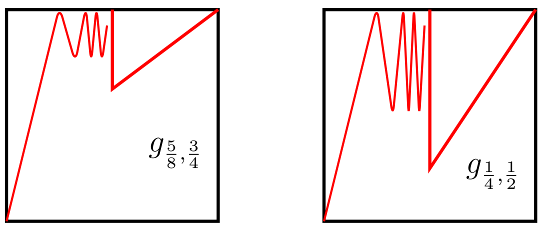

Figure 2. The graphs of $g_{\frac{5}{8},\frac{3}{4}}$ and $g_{\frac{1}{4},\frac{1}{2}}$

1. $Y_{a,c}$ is chainable if, for all $n \geq 0$, $g^n_{a,c}(a) \neq \frac{1}{2}$ and $g^n_{a,c}(c) \neq \frac{1}{2}$.

For a down fold at $(\frac{1}{2}, 1)$, and each down fold in the sequence of down folds at $\{(x_i, 1)\}$, we see that the full orbit of 1 is the constant $\{1\}$ sequence. For an up fold at some (y_i, c), by hypothesis, the full orbit of c consists of singletons. For an up fold at $(\frac{1}{2}, a)$, by hypothesis, the full orbit of a also consists of singletons. By Corollary 1(a), $Y_{a,c}$ is chainable.

By placing additional restrictions on some of the members of this family of functions, one can ensure that, for all $n \geq 0$, $g^n_{a,c}(a) \neq \frac{1}{2}$ and $g^n_{a,c}(c) \neq \frac{1}{2}$. For example, for $0 < a < c < \frac{1}{2}$, let $y_1 = a$, and let $g_{a,c}(c) = c$. We have that the full orbit of each of a and c is the constant $\{c\}$ sequence.

2. $Y_{a,c}$ is not chainable if either $a = \frac{1}{2}$ or $c = \frac{1}{2}$.

(a) Suppose $a = \frac{1}{2}$. Then we consider the one-sided up fold V of $g_{\frac{1}{2},c}$ at $(\frac{1}{2}, \frac{1}{2})$ where $c_2(V) = [\frac{1}{2}, 1]$ in the second factor space. Now, $g_{\frac{1}{2},c}$ is not right cohesive at $\frac{1}{2}$ into the first factor space. So, by Theorem 1(3), $Y_{a,c}$ is not chainable.

(b) Suppose $c = \frac{1}{2}$. Let V be the up fold of $g_{a,\frac{1}{2}}$ at $(\frac{1}{2}, y_1)$ such that $c_2(V) = [\frac{1}{2}, 1]$ in the second factor space. Again, $g_{a,\frac{1}{2}}$ is not right cohesive at $\frac{1}{2}$. So, by Theorem 1(3), $Y_{a,\frac{1}{2}}$ is not chainable.

In Example 3, we give an example of an inverse sequence where a restricted right cohesive subgraph in the graph of the second bonding function ℓ_2 produces a triod in the partial graph G_1^3, making the inverse limit not chainable. It can be checked that no up or down fold Λ of ℓ_2 produces a triod in G_1^3 since Λ does not compose to a value where ℓ_1 is not cohesive. So, condition (4) in Theorem 1 is necessary in the sense that it cannot be omitted.

Example 3. We define an inverse sequence $\{[0,1], \ell_i\}$ with limit Z. The bonding functions ℓ_1 and ℓ_2 cause Z to not be chainable.

Define ℓ_2 as follows. Let $G(\ell_2|_{[0,\frac{1}{2}]})$ be the graph of a $\sin\frac{1}{x}$-curve with endpoint $(0,0)$ and limit bar $\{\frac{1}{2}\} \times [\frac{1}{4}, \frac{3}{4}]$. Let $G(\ell_2|_{[\frac{1}{2},1]})$ be the graph of a $\sin\frac{1}{x}$-curve with the same limit bar. For the $\sin\frac{1}{x}$-curve to the left of the limit bar, we assume the down folds all have second coordinates larger than $\frac{3}{4}$, and the up folds all have second coordinates less than $\frac{1}{4}$ and larger than 0. For the $\sin\frac{1}{x}$-curve to the right of the limit bar, we assume the down folds all have second coordinates less than $\frac{3}{4}$, and the up folds all have second coordinates larger than $\frac{1}{4}$.

Define ℓ_1 as follows. Let $\ell_1(0) = [0, \frac{5}{8}]$, $\ell_1(t) = \frac{5}{8}$ for $0 < t < \frac{1}{4}$, $\ell_1(\frac{1}{4}) = [\frac{1}{2}, \frac{5}{8}]$, $\ell_1(t) = \frac{1}{2}$ for $\frac{1}{4} < t < \frac{3}{4}$, $\ell_1(\frac{3}{4}) = [\frac{1}{2}, \frac{3}{4}]$, and $\ell_1(t) = t$ for $\frac{3}{4} < t \leq 1$.

For $i > 2$, let ℓ_i be the identity mapping on $[0,1]$. Graphs of ℓ_1 and ℓ_2 are shown in Figure 3.

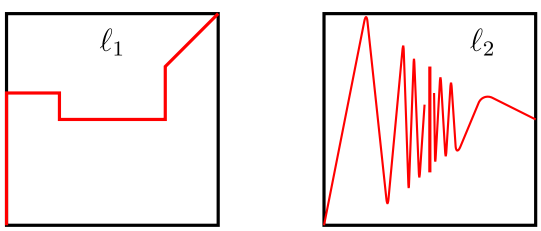

Figure 3. The graphs of ℓ_1 and ℓ_2

The inverse limit Z is not chainable. We observe that $G(\ell_2)$ has a restricted right cohesive subgraph at $\frac{1}{2}$; in particular, $\ell_2(\frac{1}{2}) = \ell_2([\frac{1}{2}, 1]) = [\frac{1}{4}, \frac{3}{4}]$. Also, ℓ_1 is not right cohesive at $\frac{1}{4}$ and not left cohesive at $\frac{3}{4}$. By either Theorem 1(4) or Theorem 4(2), this produces a triod in G_1^3. Hence, Z is not chainable.

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