A short course on groups of finite Morley rank
—Part 1—

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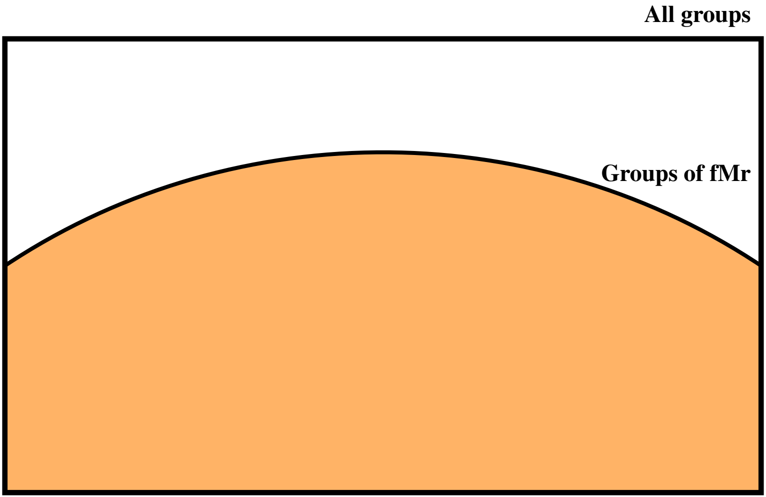
Groups of finite Morley rank (fMr)



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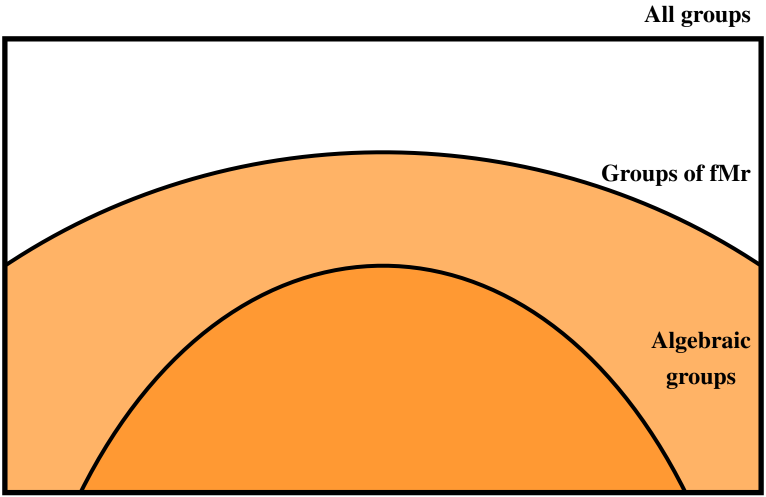

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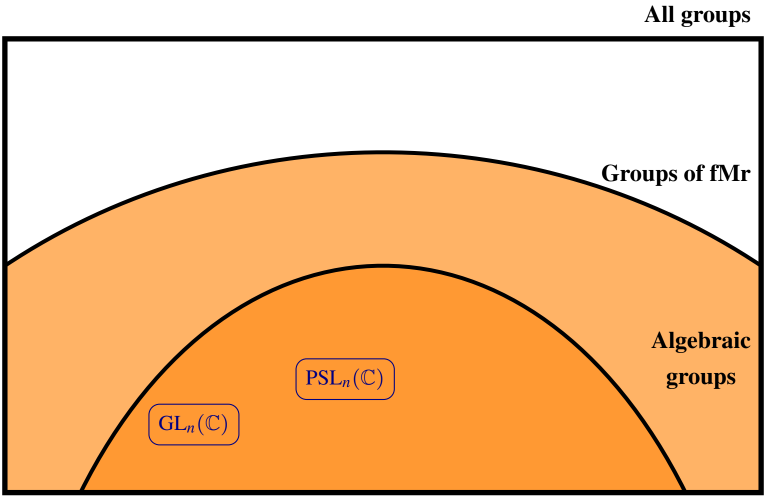

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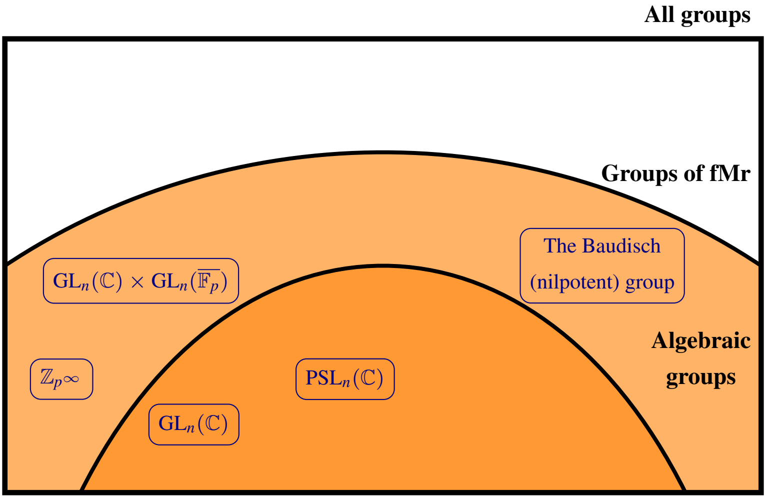

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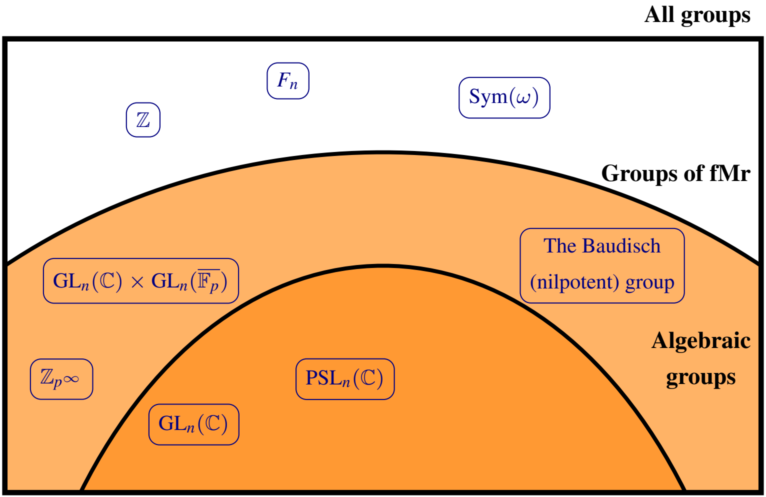
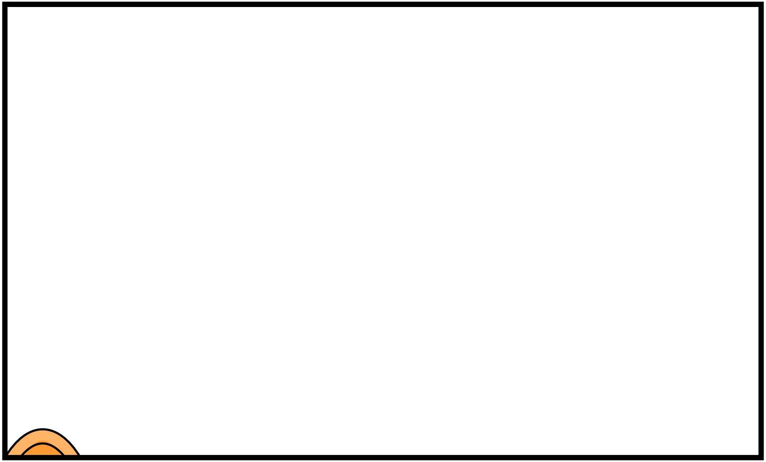


The world

All groups



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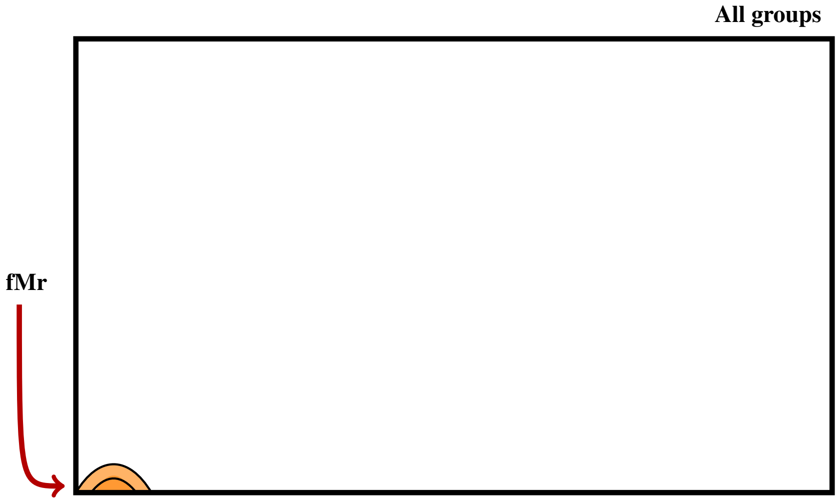
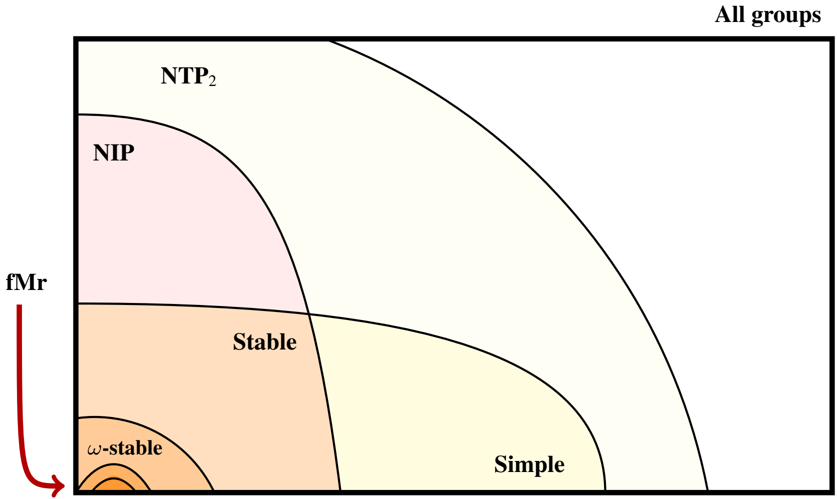
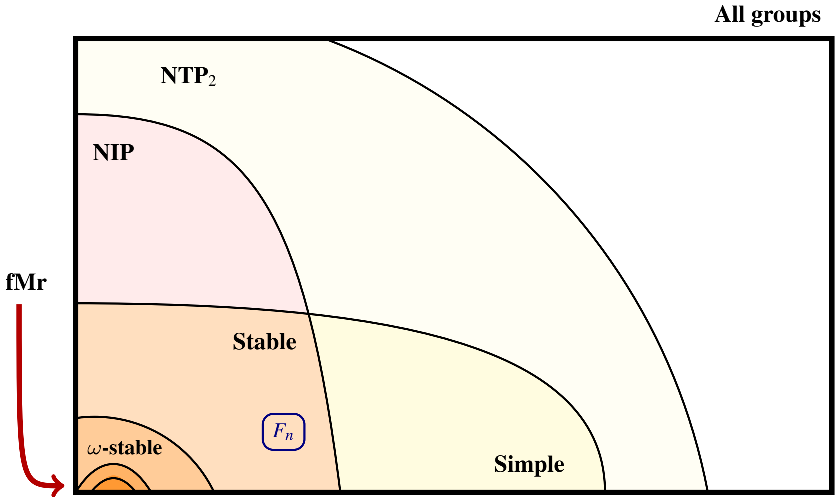
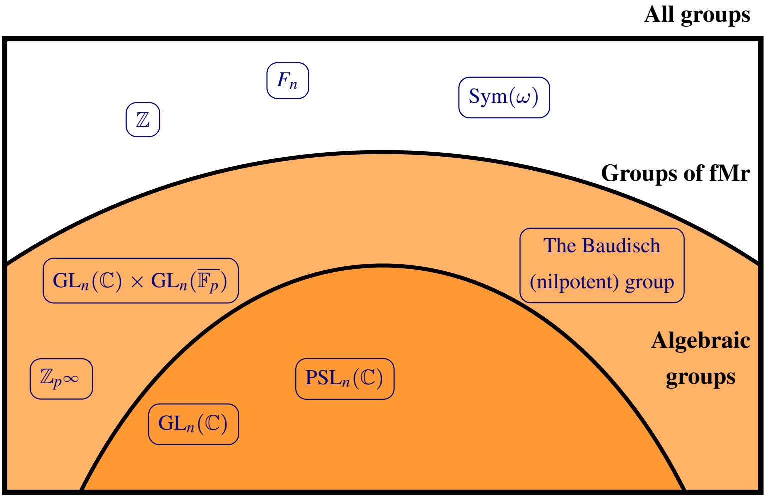



All groups
NTP$_2$
NIP
fMr
Stable
ω-stable
Simple

The world



Algebraicity Conjecture



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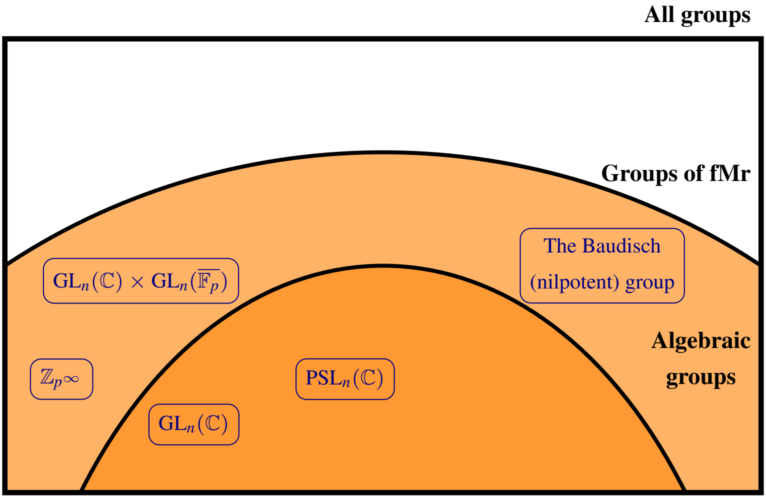

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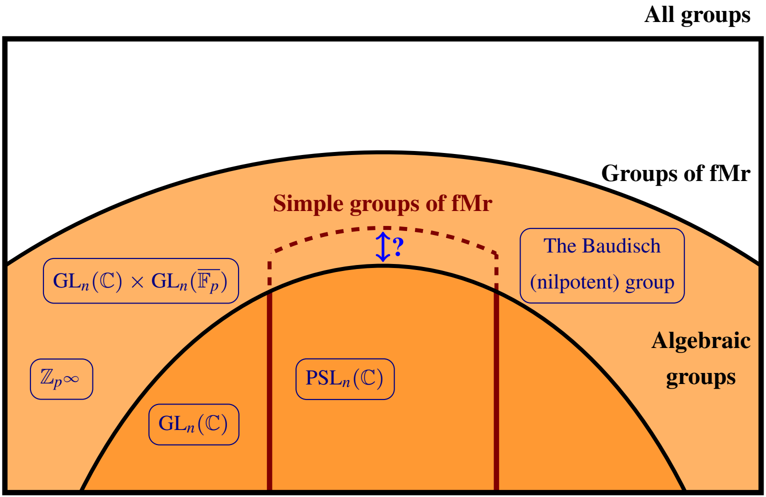

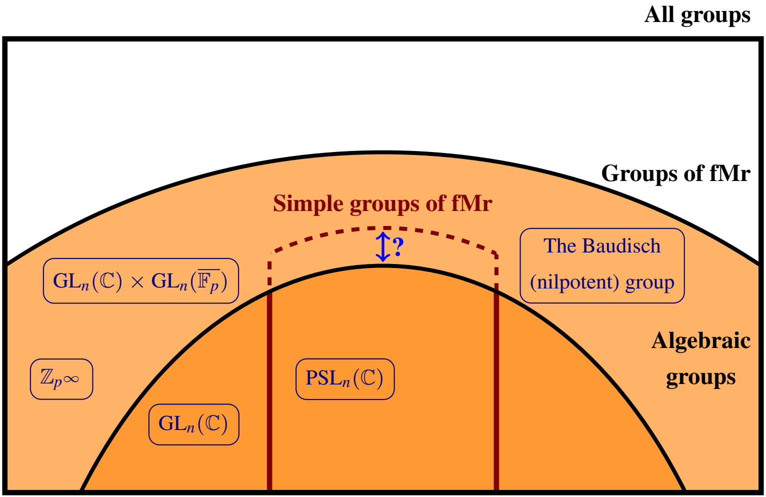

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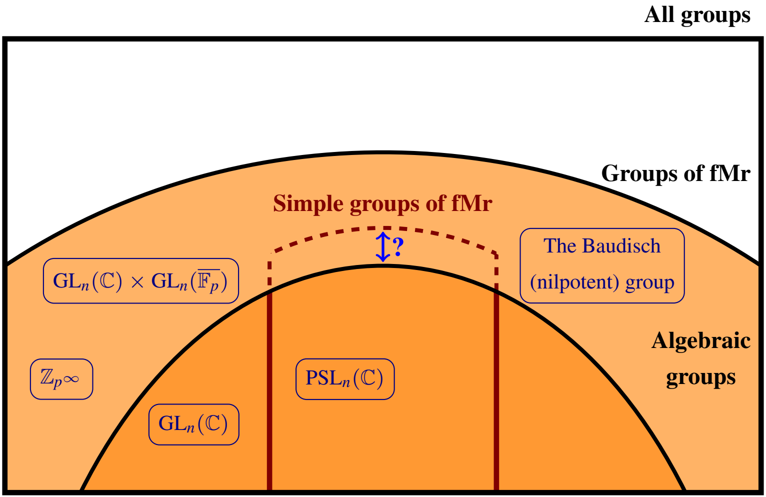

Algebraicity Conjecture: the gap, ↕ , does not exist.

The Plan

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Act I First principles, general theory, and optimism

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Act II Obstructions and pessimism

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Act I	First principles, general theory, and optimism
Act II	Obstructions and pessimism
Act III	Permutation groups

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Companion Notes

webpages.csus.edu/wiscons/research/GFMR-Minicourse-Notes.pdf

Act I

First principles, general theory, and optimism

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Thus, $G/Z(G)$ is G-definable… the underlying set and the group operations!

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Fact

If $\mathbb{K}$ is an algebraically closed field (considered as a $\mathcal{L}_{\text{RING}}$-structure), then every $\mathbb{K}$-definable set is (in definable bijection with) a constructible set.

Ranked structures

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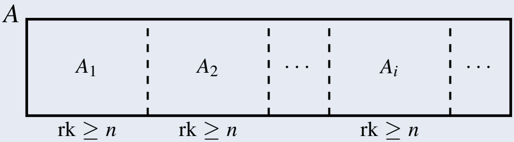

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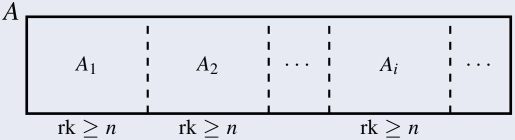

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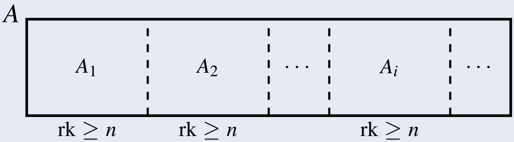

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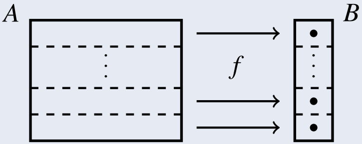

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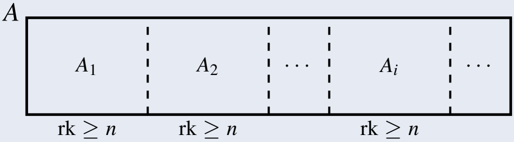

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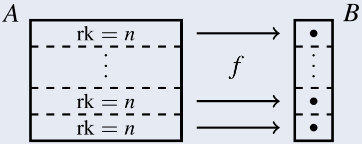

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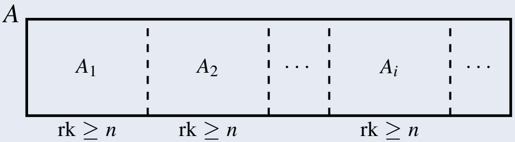

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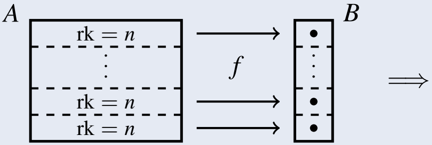

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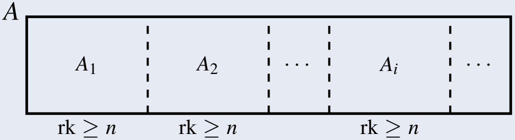

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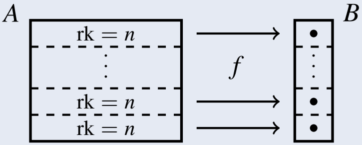

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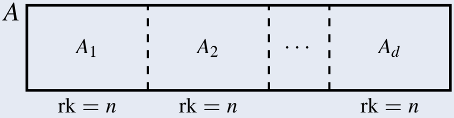

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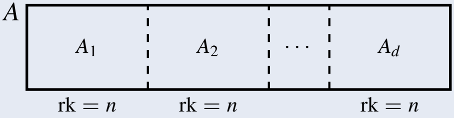

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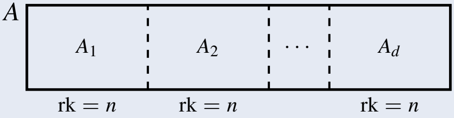

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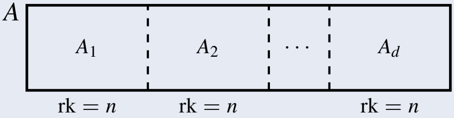

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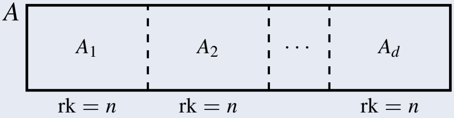

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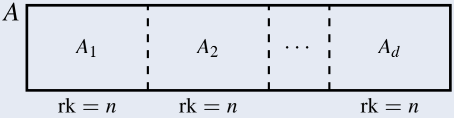

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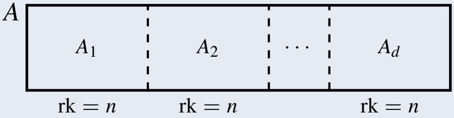

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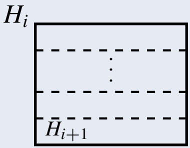

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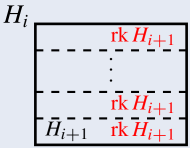

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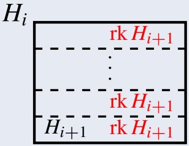

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$\mathbb{Z}$ does NOT have fMr because $\mathbb{Z} > 2\mathbb{Z} > 4\mathbb{Z} > 8\mathbb{Z} > \cdots$,

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- *(Definable Hull) Every subgroup H is contained in a minimal definable subgroup $\mathrm{d}(H)$.*

Quick check-in

Exercise

Suppose that G is a connected group of fMr. Show that if $a \in G$ has finitely many conjugates, then $a \in Z(G)$.

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Algebraic analogies: semisimplicity (kind of)

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- Thus, every $a \in T_0$ has a finitely many N-conjugates, so $a \in Z(N)$ □

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Fact (Conjugacy of Maximal Tori, Cherlin—2005)

Any two maximal decent tori of a group of fMr are conjugate.

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Definition

Let p be a prime. A definable subgroup of a group of fMr G is called p-unipotent if it is a connected nilpotent p-group of *bounded exponent*.

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Let p be a prime. If G is a connected group of fMr with no nontrivial p-unipotent subgroup, then every p-element of G is contained in a p-torus.

Algebraic analogies: Borel subgroups

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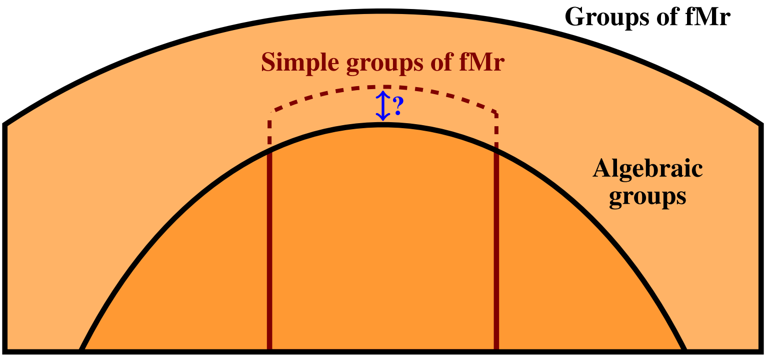

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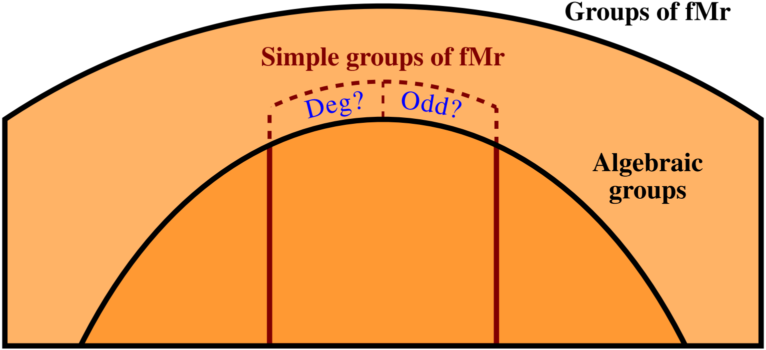

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Fact (Frécon—2018)

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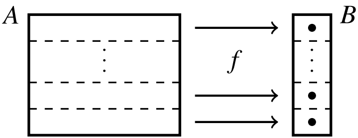

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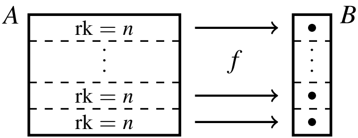

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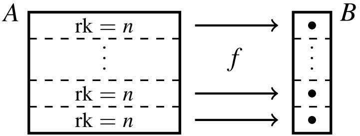

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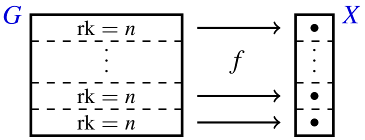

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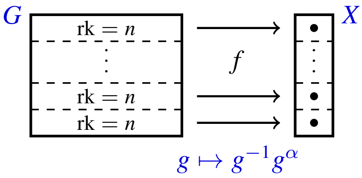

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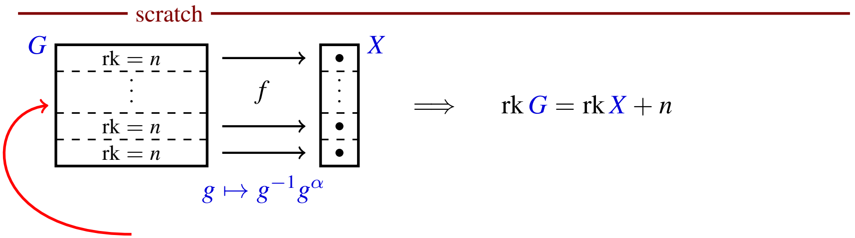

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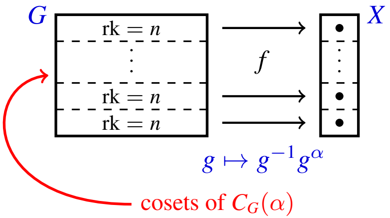

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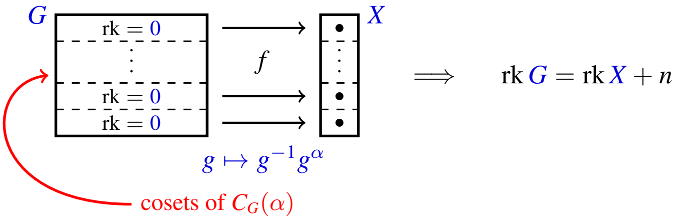

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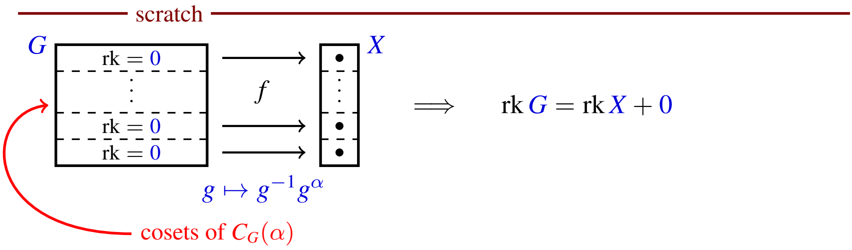

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- $X \subseteq Z(G)$

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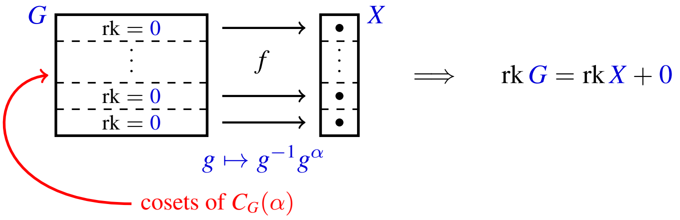

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Let G be a *connected* group of fMr.

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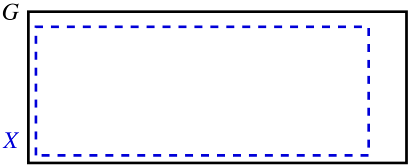

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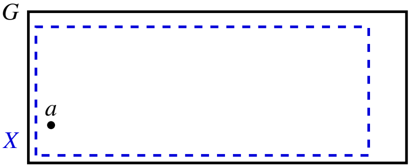

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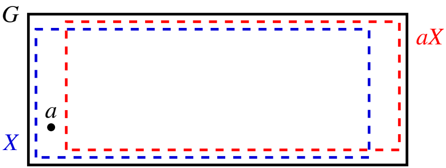

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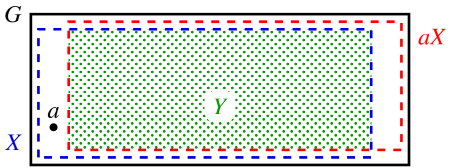

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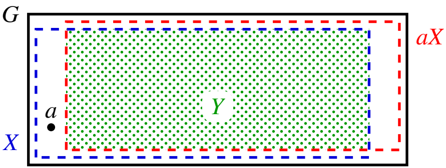

$= (ax)^{\alpha} =$

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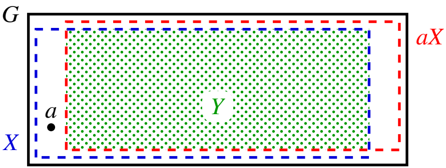

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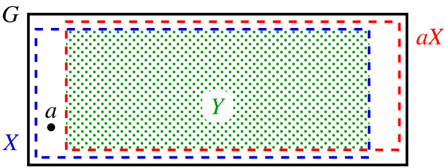

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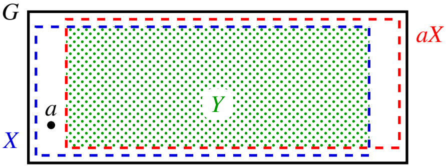

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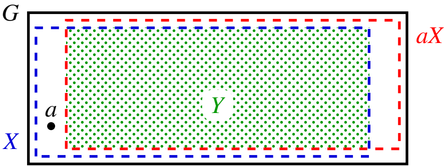

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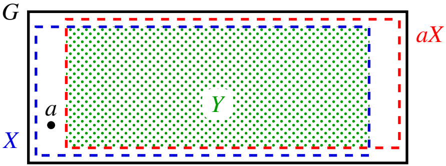

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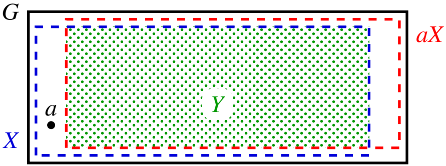

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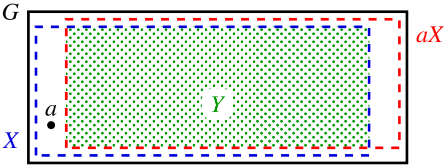

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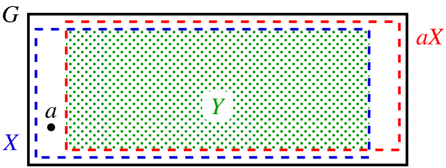
- $X \subseteq Z(G) \implies Z(G)$ is generic in G

scratch



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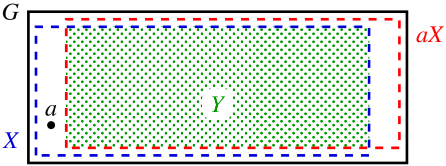
- G is abelian

scratch



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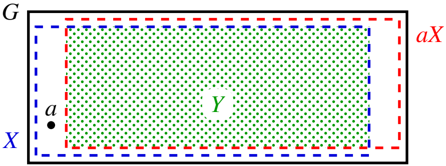
- G is abelian $\implies X$ is a generic subgroup

scratch



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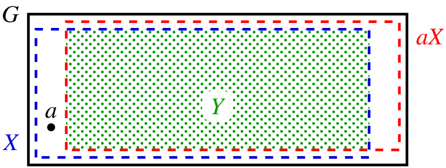
scratch



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The End—Thank You