Representations of Sym(n) of minimal dimension

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Joint work with Luis Jaime Corredor (Bogotá) and Adrien Deloro (Paris)

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Representations and Modules

A first example

A first example

Example

What group is this?

$$G = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

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Remark

Remember that matrices can/should be viewed as functions acting on the vector space.

A first example

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- Here, each $A \in G$ is associated with $F^2 \to F^2 : \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \mapsto A \cdot \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$.

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- Here, each $A \in G$ is associated with $F^2 \to F^2 : \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \mapsto A \cdot \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$.
- To understand a matrix this way, it's enough to understand what it does to a basis.
- The standard basis: $e_1 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] = x$-axis, $e_2 = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] = y$-axis

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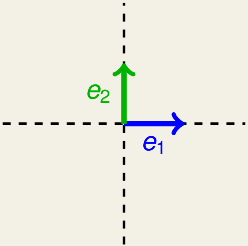

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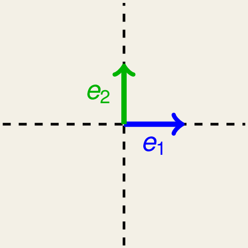

- Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

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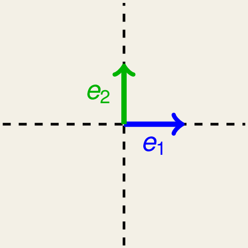

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$$A \cdot e_1 = A \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$$

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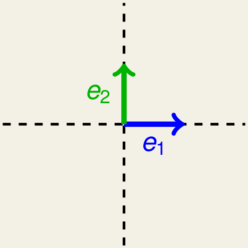

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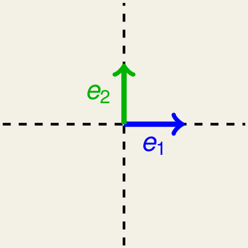

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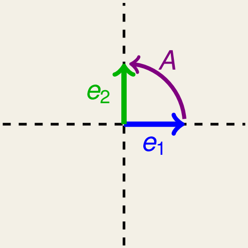

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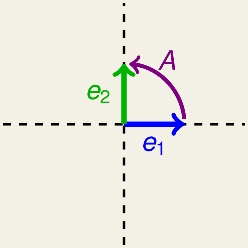

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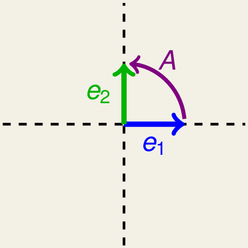

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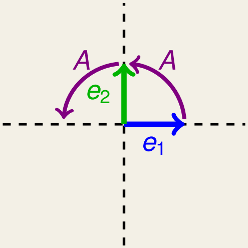

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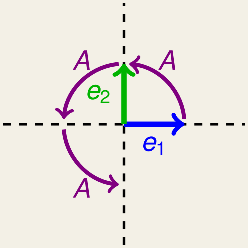

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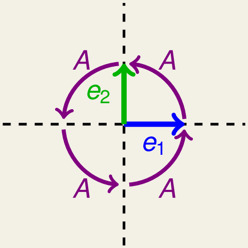

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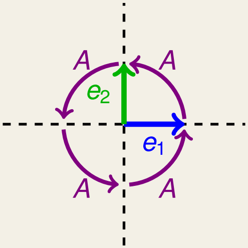

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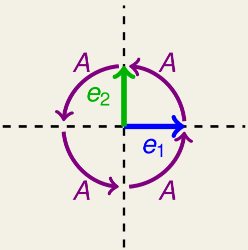

- Let $A = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$.

  A acts as a 90° rotation

A first example

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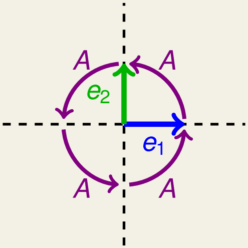

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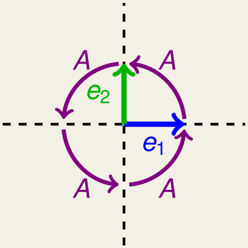

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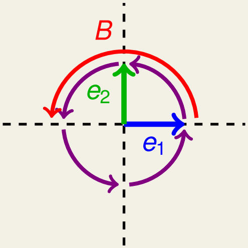

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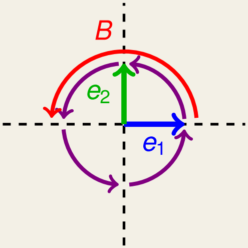

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$$\begin{aligned} B \cdot e_1 &= B \cdot \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right] = -e_1 \\ B \cdot e_2 &= B \cdot \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right] = -e_2 \end{aligned}$$

A first example

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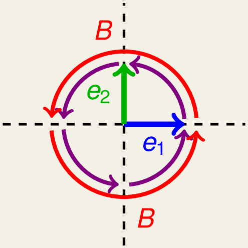

- Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

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- Let $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

$$B \cdot e_1 = B \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} = -e_1$$
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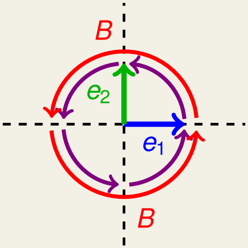

- Let $A = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$.

  A acts as a 90° rotation

- Let $B = \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$.

  B acts as a 180° rotation

A first example

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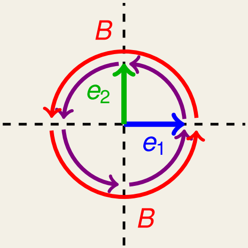

- Let $A = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$.

  A acts as a 90° rotation

- Let $B = \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$.

  B acts as a 180° rotation

- Let $C = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$.

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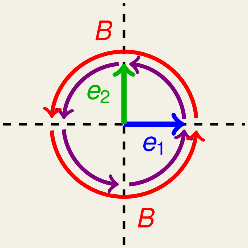

- Let $A = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$.

  A acts as a 90° rotation

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  B acts as a 180° rotation

- Let $C = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$.

  C acts as a 270° rotation

A first example

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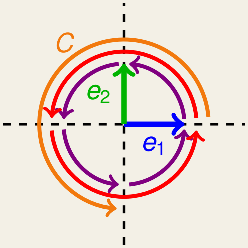

- Let $A = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$.

  A acts as a 90° rotation

- Let $B = \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$.

  B acts as a 180° rotation

- Let $C = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$.

  C acts as a 270° rotation

A first example

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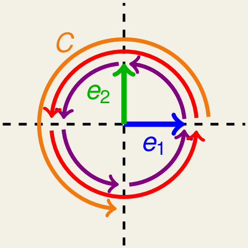

- Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

  A acts as a 90° rotation

- Let $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

  B acts as a 180° rotation

- Let $C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

  C acts as a 270° rotation

Representations

Definition

A (linear) representation of G is a homomorphism $\rho : G \to GL_n(F)$, where $\mathrm{GL}_n(F)$ denotes the invertible $n \times n$ matrices with entries in F.

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Example

Let $C_4 = \{1, r, r^2, r^3\}$ be cyclic of order 4. Then

$$1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad r^2 \mapsto \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$r \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad r^3 \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

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- $\rho : G \to V$ is a representation iff $\rho(g) \cdot v$ makes V a G-module.

Revisiting the first example

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Let $C_4 = \{1, r, r^2, r^3\}$ be cyclic of order 4. Then

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Sym(n)-representations and modules

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- We will explore the first three a bit in the context of $\mathrm{Sym}(n)$.

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We call this the natural permutation module, denoted perm_F^n.

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The corresponding representation is

$$(12) \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (123) \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

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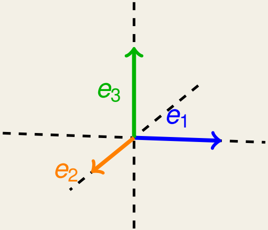

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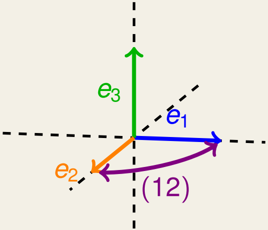

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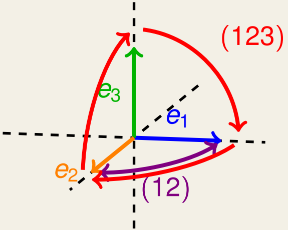

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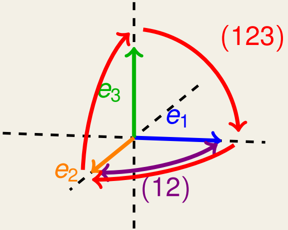

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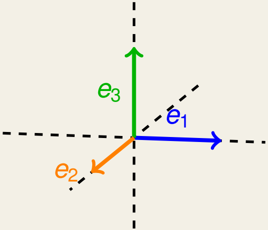

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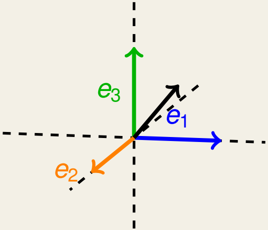

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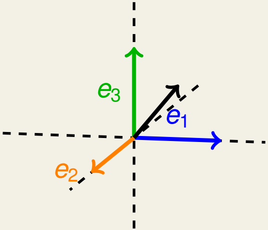

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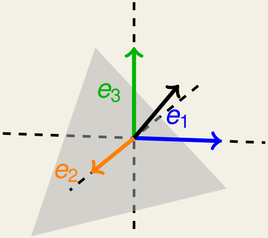

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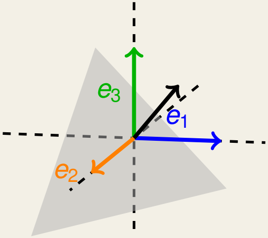

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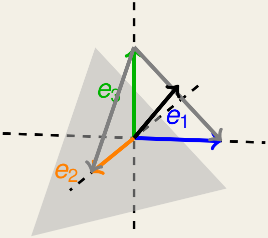

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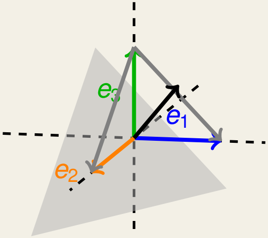

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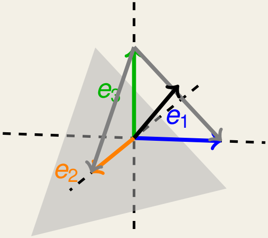

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- These give rise to submodules.

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Definition (Standard Module)

Let $V = F^n$ with basis $e_1, e_2, \ldots, e_n$. Define the standard module to be the submodule of perm_F^n defined as

$$\begin{aligned}\text{std}_F^n &= \{\text{vectors whose coordinates sum to zero}\} = H \\ &= \langle \underbrace{e_1 - e_n}_{f_1}, \underbrace{e_2 - e_n}_{f_2}, \ldots, \underbrace{e_{n-1} - e_n}_{f_{n-1}} \rangle.\end{aligned}$$

Remark

A G-module (or representation) is called faithful if no nontrivial element of G acts like the identity.

- perm_F^n is a faithful Sym(n)-module of dimension n.
- std_F^n is a faithful Sym(n)-module of dimension $n - 1$.
- The submodule $Z = \langle e_1 + \cdots + e_n \rangle$ is not faithful.

The standard module for Sym(3)

Example

Let's look at $\mathrm{std}_F^3 = \langle f_1, f_2 \rangle$ where $f_1 = e_1 - e_3$ and $f_2 = e_2 - e_3$. The module structure is given by

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Is there a faithful Sym(3)-module of even smaller dimension?

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When char $F \mid n$, we define the reduced standard module to be the quotient $\overline{\mathrm{std}}_F^n = \mathrm{std}_F^n / \langle e_1 + \cdots + e_n \rangle$.

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The reduced standard module for Sym(5)

Example

Let $F = \mathbb{Z}/5\mathbb{Z}$. Let's look at $\overline{\text{std}}_F^5 = \langle f_1, f_2, f_3, f_4 \rangle / \langle e_1 + \cdots + e_5 \rangle$.

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The reduced standard module for Sym(5)

Example

Let $F = \mathbb{Z}/5\mathbb{Z}$. Let's look at $\overline{\mathrm{std}}_F^5 = \langle f_1, f_2, f_3, f_4 \rangle / \langle e_1 + \cdots + e_5 \rangle$.

- $f_1 + f_2 + f_3 + f_4 = e_1 + e_2 + e_3 + e_4 + e_5 \equiv 0$
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Irreducibility of the (reduced) standard module

Question

Is there a faithful Sym(5)-module of even smaller dimension?

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Remark

This only says $\overline{\mathrm{std}}_F^n$ (char $F \mid n$) and std_F^n (char $F \nmid n$) can't be "reduced" further. It doesn't say smaller modules can't be found other ways.

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  - E.g. Morley rank, o-minimal dimension, Prüfer p-rank

A new context: *G*-modules with an additive dimension

Modules with an additive dimension

Definition

We redefine a G-module to be a vector space V together with a multiplication $g \cdot v$ satisfying the axioms from before.

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We redefine a G-module to be a abelian group V together with a multiplication $g \cdot v$ satisfying the axioms from before. Additionally,

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- By dim-connectedness of V, $V = B + C$ □

Sym(n)-modules of minimal dimension

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Let $\mathrm{perm}_{\mathbb{Z}}^{n}$ be defined analogously to before.

The standard module for Sym(n) — CDW-remix

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These all carry a Sym(n)-multiplication as before.

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Suppose V is faithful and dim-irreducible $\mathrm{Sym}(n)$*-module with* $\mathrm{char}\, V = q$ *and* $\dim V = d < n$*.*

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The same is true for $\mathrm{Alt}(n)$*-modules provided* $n \geq 10$ *when* $q = 2$*.*

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  - The geometric condition on $B_V((12))$ leads to recognition of std_L^n or $\overline{\mathrm{std}}_L^n$ with $L = B_V((12))$

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Love groups? Want to learn more about representations/modules in the classical sense? In the new context? Want to start exploring some of these questions? There will be a focused series of talks in Spring 2021 on this topic. Email joshua.wiscons@csus.edu for info.

Thank You