The minimal faithful Sym(n)- and Alt(n)-modules

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Logic Seminar

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Joint work with Luis Jaime Corredor (Bogotá) and Adrien Deloro (Paris)

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Freedom (and passport) for Tuna Altınel

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Outline

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Initial context and motivation

- Groups of finite Morley rank
- Connections to high degrees of generic transitivity

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New context and results

- Modules with an additive dimension
- The faithful Sym(n)- and Alt(n)-modules of minimal dimension

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Reflections and lingering questions

Initial context and motivation (and distractions)

Groups of finite Morley rank

Groups of finite Morley rank: definition

Definition

A structure $\mathcal{M}$ is ranked if its universe of definable (and interpretable) sets carries a well-behaved notion of dimension $\text{rk} : \mathcal{U}_{\text{DEF}}(\mathcal{M}) \to \mathbb{N}$, analogous to Zariski dimension.

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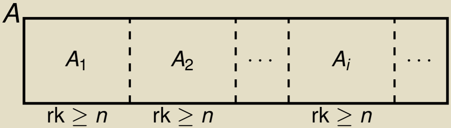
A

A_1 A_2 $\cdots$ A_i $\cdots$

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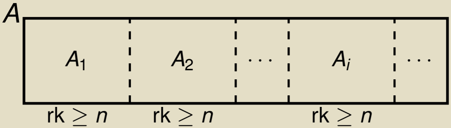

- (*Additivity*) If $f : A \twoheadrightarrow B$ is definable with fibers of constant rank n, then $\mathrm{rk}(A) = \mathrm{rk}(B) + n$.

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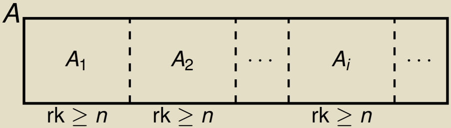

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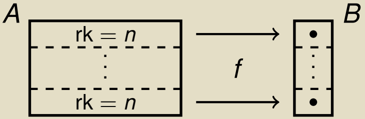

A

$\mathrm{rk} = n$

$\vdots$

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f

B

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Theorem (Poizat)

A group is ranked $\iff$ *it is a group of finite Morley rank.*

Groups of finite Morley rank: examples

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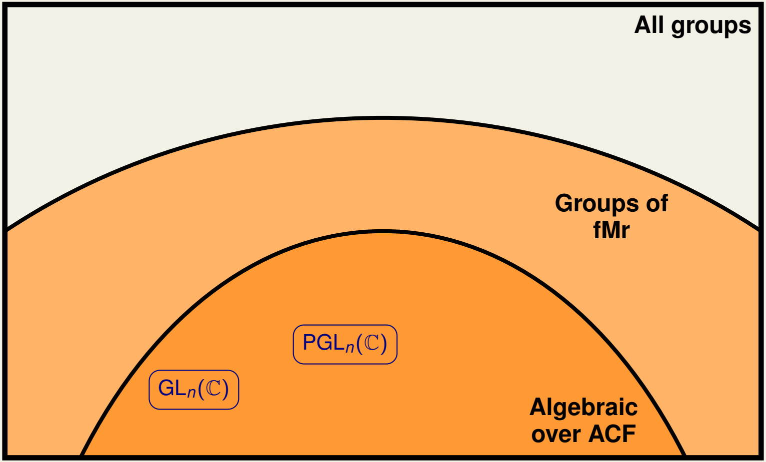
5. Groups definable from a structure of fMr
6. Algebraic groups over algebraically closed fields: $\mathrm{GL}_n(\mathbb{K})$, $\mathrm{PGL}_n(\mathbb{K})$, $\ldots$

Groups of finite Morley rank: landscape



All groups

Groups of fMr

Algebraic over ACF

$\mathrm{PGL}_n(\mathbb{C})$

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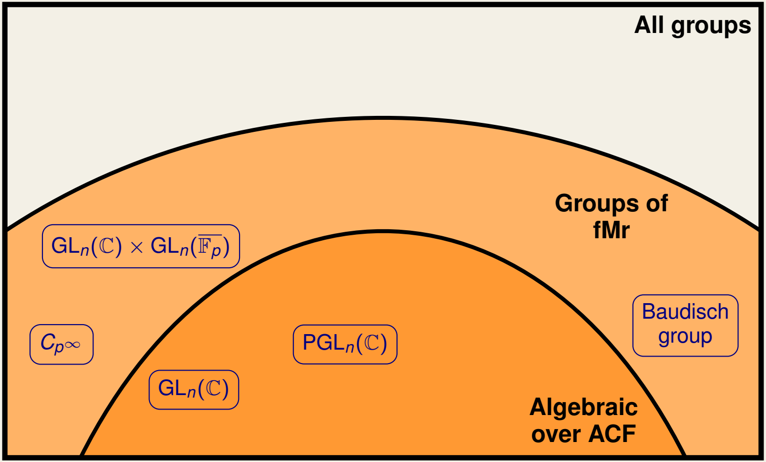
Groups of finite Morley rank: landscape



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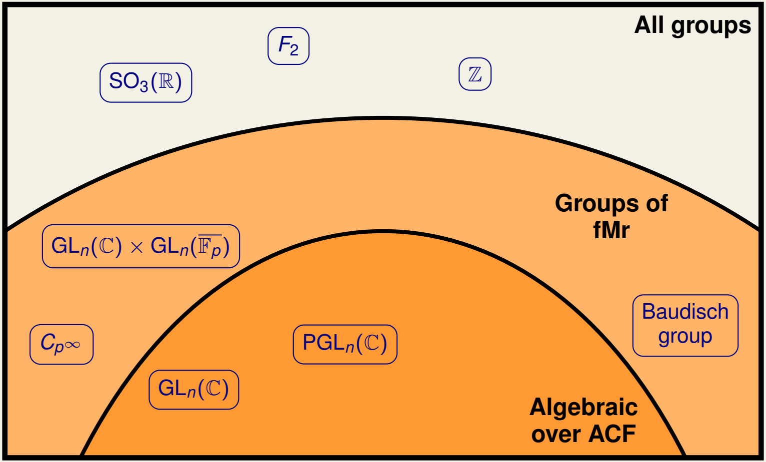

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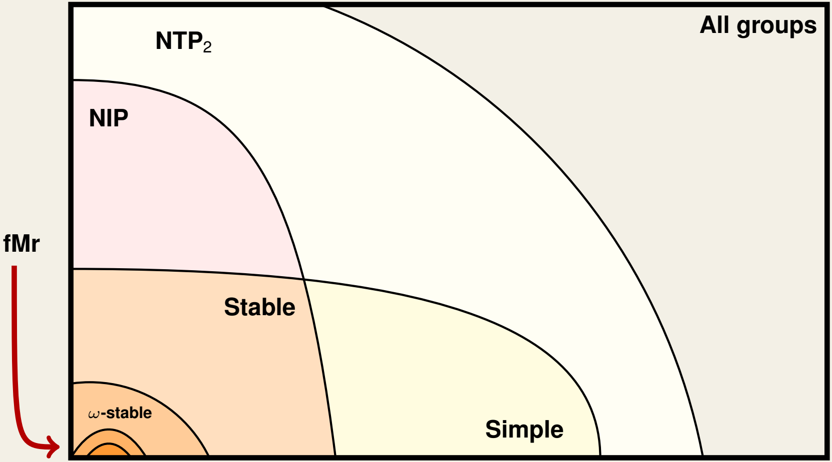

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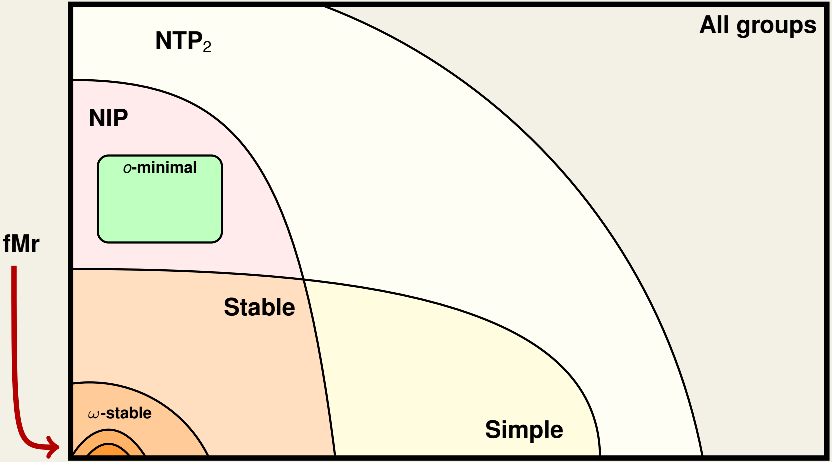

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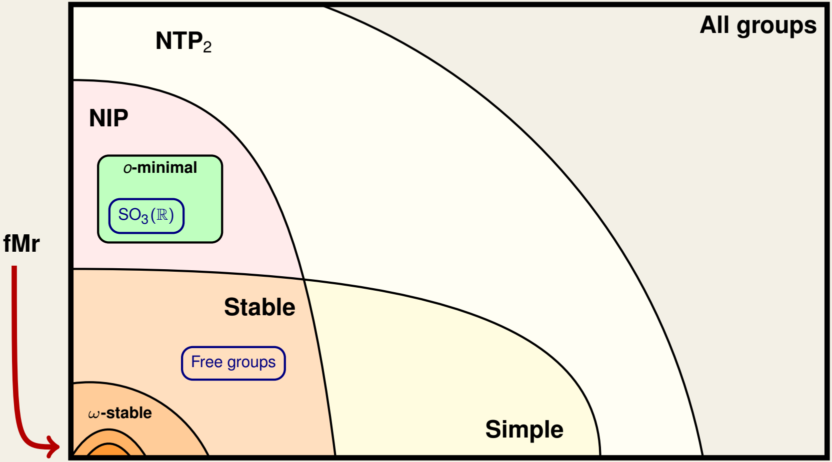
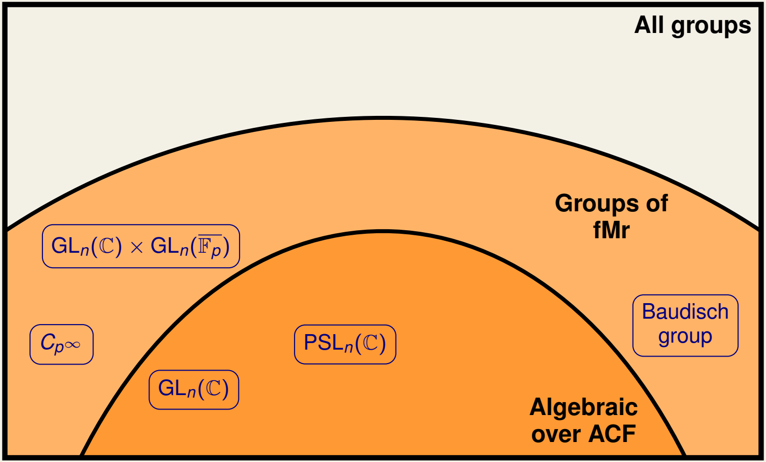


An aside: the Algebraicity Conjecture



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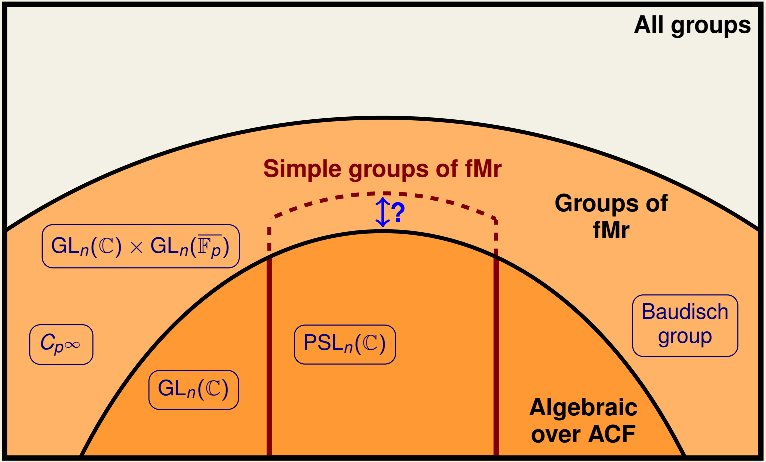

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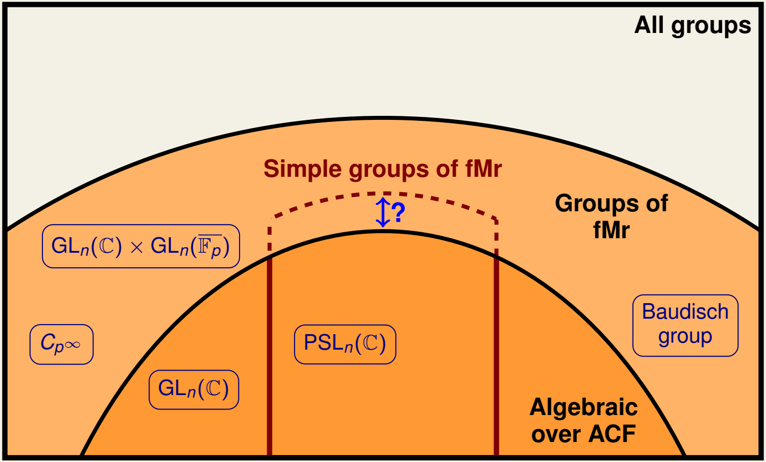

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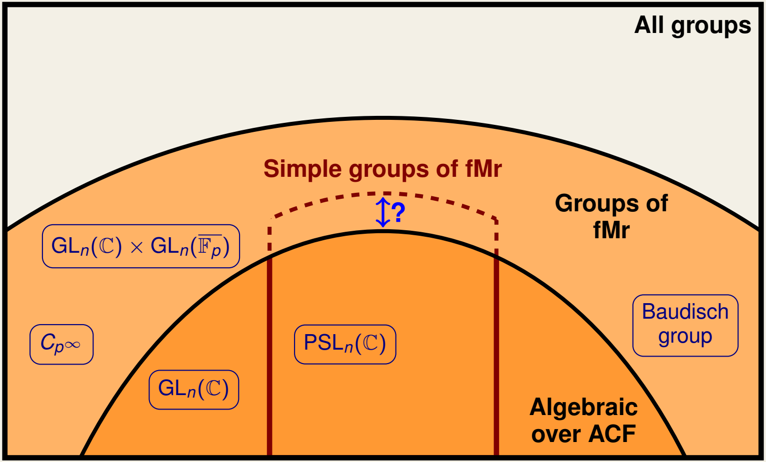

Algebraicity Conjecture: the gap, ↕ , does not exist.

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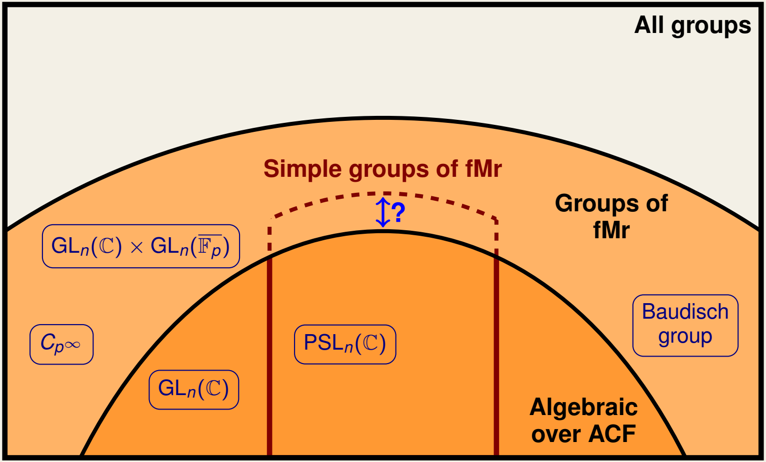

Algebraicity Conjecture: every simple group of fMr is algebraic over an ACF.

Initial context and motivation (and distractions)

Permutation groups and generic transitivity

Generic *n*-transitivity

Definition

Let $G \curvearrowright X$ be a permutation group of fMr. The action is generically *n*-transitive if there is an orbit $\mathcal{O} \subset X^n$ with $\text{rk}(X^n - \mathcal{O}) < \text{rk}(X^n)$.

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- $\mathcal{O}$ is the set of bases of K^n: orbit of $(e_1, \ldots, e_n)$

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Assume that $G_1 \curvearrowright X_1$ and $G_2 \curvearrowright X_2$ are both generically n-transitive.

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- $G_1 \times G_2 \curvearrowright X_1 \times X_2$ is generically n-transitive

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- $G_1 \times G_2 \curvearrowright X_1 \times X_2$ is generically n-transitive
- $\mathcal{O} = \mathcal{O}_1 \times \mathcal{O}_2$

Limits to generic n-transitivity

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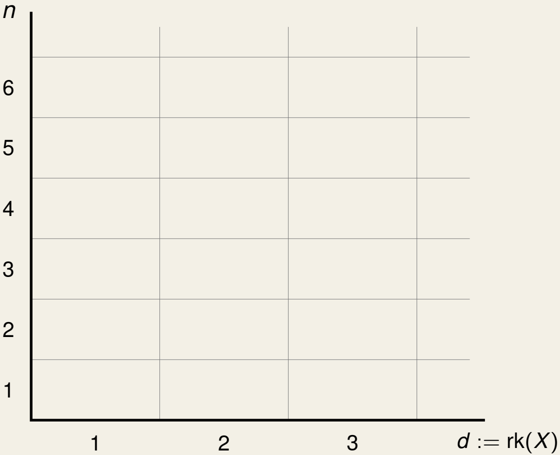

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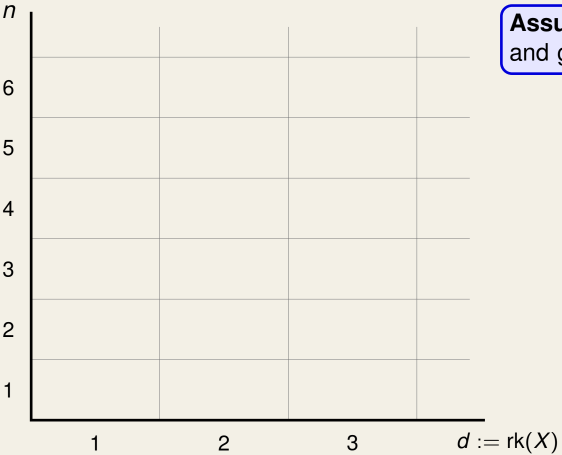
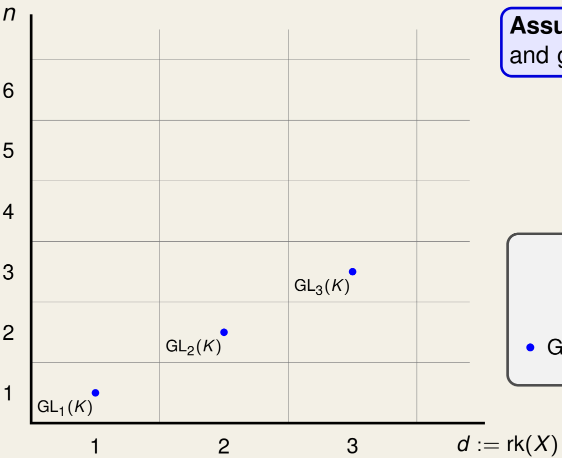


Assume: $G \curvearrowright X$ is transitive and generically n-transitive

Limits to generic *n*-transitivity



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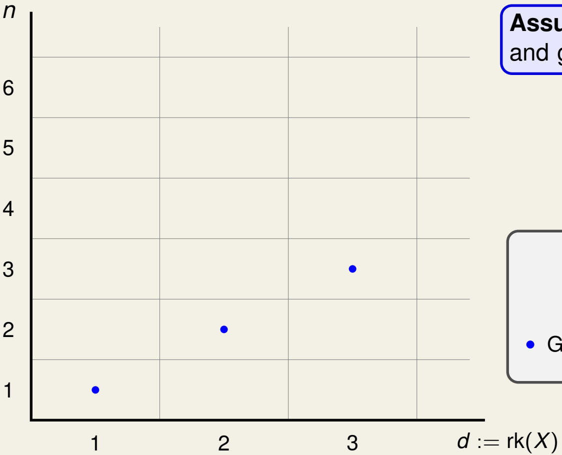

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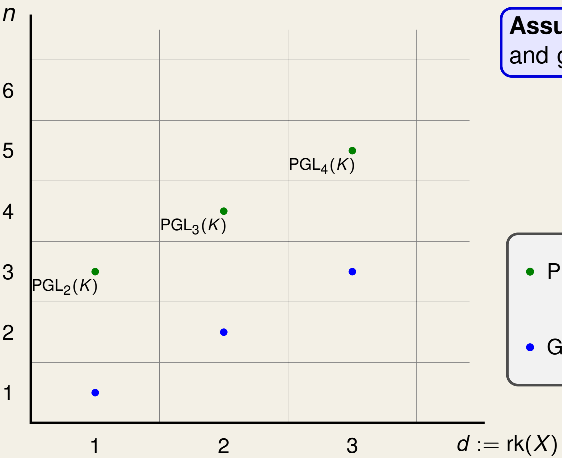

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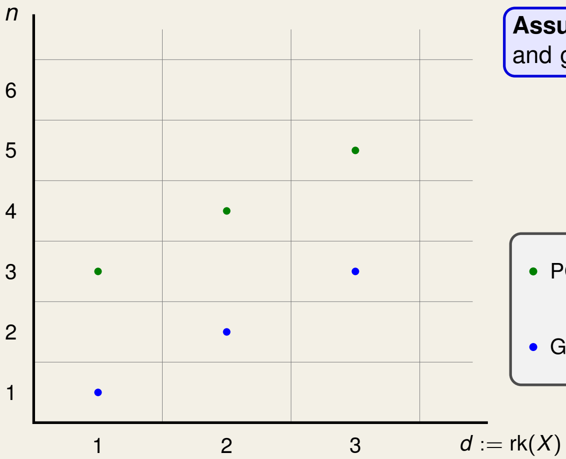

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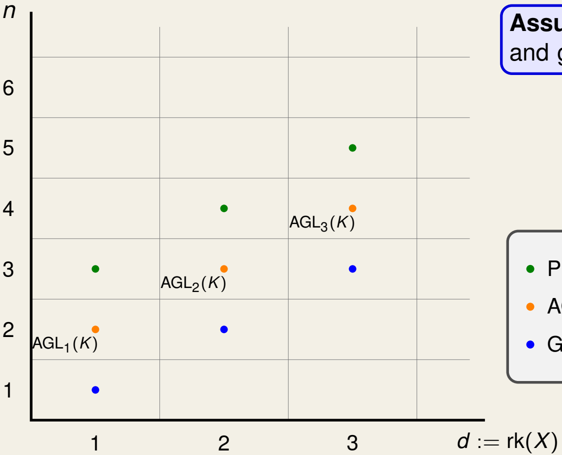

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Limits to generic *n*-transitivity

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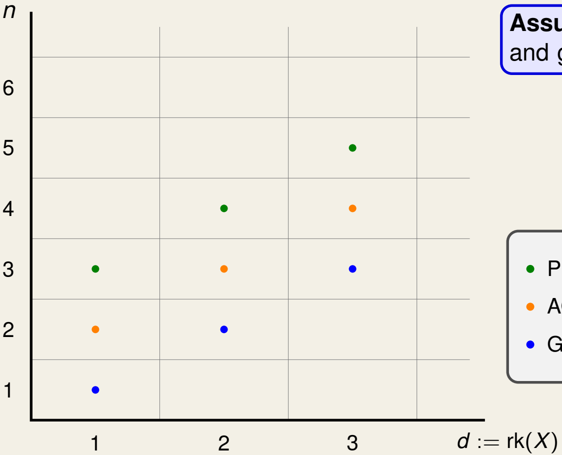

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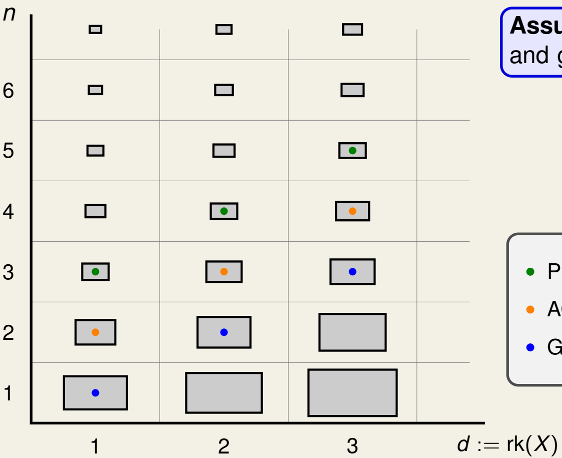

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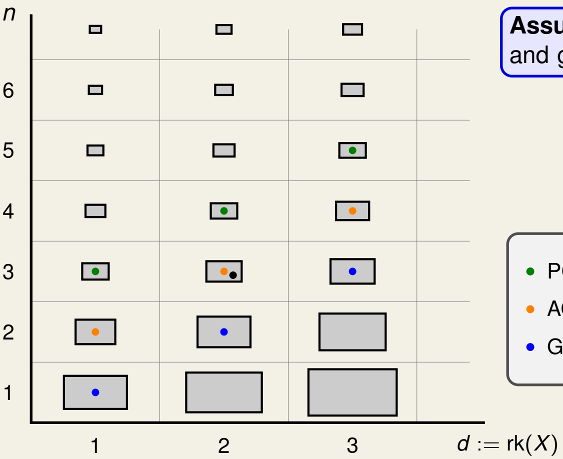

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Limits to generic n-transitivity

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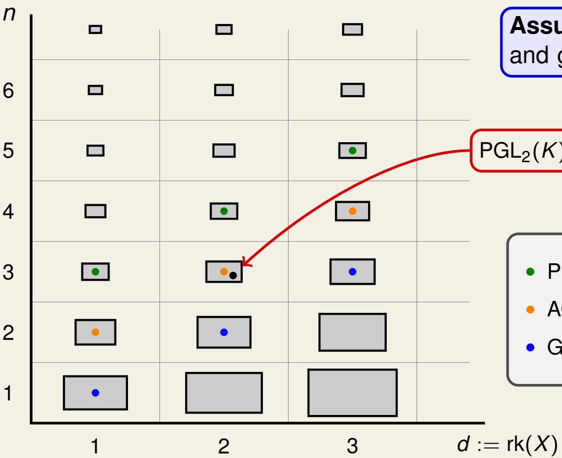

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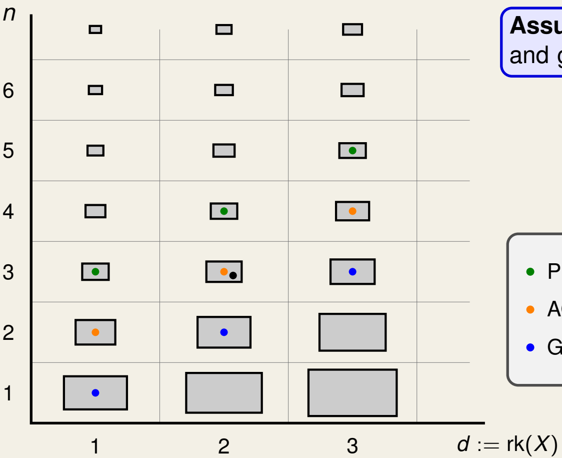

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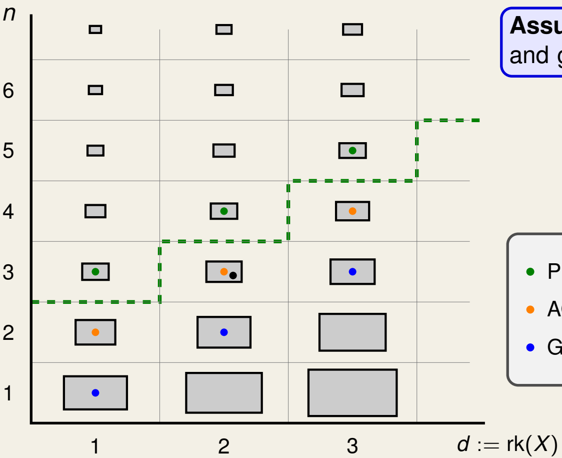

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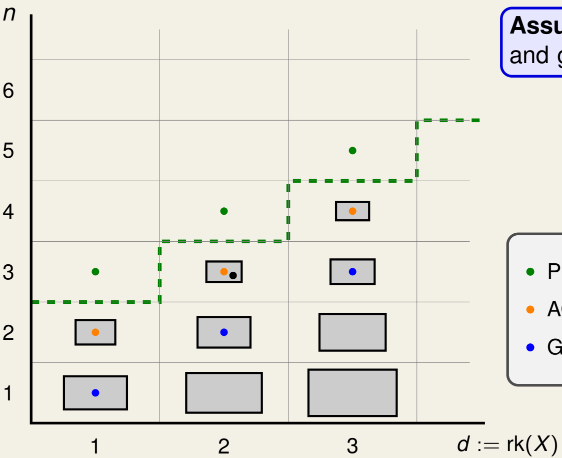

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So we turn to the study of $\mathrm{Sym}(n)$-modules (in a general context).

New context and results

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3. One could axiomatize the appropriate universe for our context, but $\mathcal{U}(V)$ is ultimately what we focus on.

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Further, if V has no proper nontrivial (dim-connected) G-modules, we say V is dim-irreducible.

A crucial parameter: the characteristic

Definition (Characteristic)

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Remark

Dim-irreducible modules always have a characteristic.

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Let V be a $\langle g\rangle$-module with $|g| = 2$. Assume $\operatorname{char} V$ exists and is not 2 (or simply V is 2-divisible). Set

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- Our hypotheses imply $\dim \Omega_2(V) = 0$ $\square$

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New context and results

The faithful Sym(n)- and Alt(n)-modules of minimal dimension

The standard module for Sym(n)

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- $\mathrm{perm}_{L}^{n} = \mathrm{perm}_{\mathbb{Z}}^{n} \otimes L = \{e_1 \otimes c_1 + \cdots + e_n \otimes c_n\}$
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The standard module for Sym(n)

Definition (Standard Module)

Let $\mathrm{perm}^n_{\mathbb{Z}} = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n$ be the Sym(n)-module where the e_i are permuted naturally. There are two obvious submodules:

- $\mathrm{triv}^n_{\mathbb{Z}} = \{ce_1 + \cdots + ce_n\}$
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For any abelian group L (with trivial Sym(n)-action), we define:

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Remark

Notice that $\mathrm{std}^n_L = \overline{\mathrm{std}}^n_L \iff \Omega_n(L) = 0$.

The standard module for Sym(n)

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$$\mathrm{perm}_{T_0}^n \cong T,\ \mathrm{std}_{T_0}^n \cong T \cap \mathrm{SL}_n(\mathbb{C}),\text{ and } \overline{\mathrm{std}}_{T_0}^n \cong \overline{T \cap \mathrm{SL}_n(\mathbb{C})} \leq \mathrm{PSL}_n(\mathbb{C}).$$

The minimal faithful Sym(n)- and Alt(n)-modules

Theorem (Corredor-Deloro-W 2018–2021)

Suppose V is faithful and dim-irreducible Sym(n)*-module with* char $V = q$ *and* $d := \dim V < n$.

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The same is true for Alt(n)*-modules provided* $n \geq 10$ *when* $q = 2$.

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4. Finally, we control the kernel. ☐

Step 2 - Extension

Extension Lemma

Let $n \geq 7$ *and* V *be a faithful and dim-irreducible* $\mathrm{Alt}(n)$*-module.*

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Moreover, up to tensoring with the signature, the extension satisfies the assumption of the Recognition Lemma.

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Moreover, up to tensoring with the signature, the extension satisfies the assumption of the Recognition Lemma.

Remark

We again say nothing about the dimension of V.

Step 3 - Geometrization

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Remark

The proof of the main theorem is readily assembled from

$$\text{Geometrization} \to \text{Extension} \to \text{Recognition}$$

with only one fairly minor remaining point to sort out.

Reflections and lingering questions

Final Thoughts

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4. What about G-modules where the “module” is nonabelian? There would be immediate applications for this to the Borovik-Cherlin Problem.

Thank You