04 – Matrix-Vector Products

Definition: Matrix-Vector Product (MVP)

Suppose that A is an $m \times n$ matrix, and let $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ be in $\mathbb{R}^n$. Let $\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_n$ be the columns of A. Then we define the product $A\mathbf{x}$ by

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n.$$

1. Compute the following.

(a) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

(c) $\begin{bmatrix} 7 & -3 \\ 2 & 1 \\ 9 & -6 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ -5 \end{bmatrix}$

Theorem

Suppose that A is an $m \times n$ matrix, and let $\mathbf{b}$ be in $\mathbb{R}^m$. Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ be the columns of A. Then each of the following have exactly the same solution sets.

- **Matrix equation:** $A\mathbf{x} = \mathbf{b}$
- **Vector equation (with columns of** A**):** $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$
- **Linear system (as an augmented matrix):** $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \mid \mathbf{b} \end{bmatrix}$

Theorem

Suppose that A is an $m \times n$ matrix. Then the following are logically equivalent. (If one is true, they all are; if one is not true, none are.)

(a) For every $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

- In other words, the system with augmented matrix $\begin{bmatrix} A \mid \mathbf{b} \end{bmatrix}$ always has a solution.

(b) For every $\mathbf{b}$ in $\mathbb{R}^m$, $\mathbf{b}$ is a linear combination of the columns of A.

(c) The columns of A span $\mathbb{R}^m$.

- That is, every vector in $\mathbb{R}^m$ is in $\text{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ where $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are the columns of A.

(d) A has a pivot position in every row.

2. Determine if $A\mathbf{x} = \mathbf{b}$ has a solution for every choice of $\mathbf{b}$ in each case below.

(a) $A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 2 & 3 \\ -5 & 6 & 11 \\ 3 & -4 & -2 \\ 3 & 0.5 & 0 \end{bmatrix}$