06 – Linear Independence

Definition: Linear Independence

Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$. The vectors are called **linearly independent** if the equation

$$x_1\mathbf{v}_1 + \cdots + x_k\mathbf{v}_k = \mathbf{0}$$

has *only one* solution (which is the trivial solution). If the equation has *more than one* solution, then the vectors are called **linearly dependent**.

1. Determine if each set of vectors is linearly independent or dependent.

(a) $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(b) $\begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix}, \begin{bmatrix} -6 \\ -10 \\ 4 \end{bmatrix}$

(c) $\begin{bmatrix} 3 \\ -5 \\ -6 \end{bmatrix}, \begin{bmatrix} -6 \\ 10 \\ 4 \end{bmatrix}$

(d) $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} -7 \\ 3 \\ 4 \end{bmatrix}$

2. Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ -3 \\ 6 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 3 \\ -9 \end{bmatrix}$. Show these three vectors are linearly dependent, and write down a dependence relation.

3. Let $\mathbf{v}_1, \mathbf{v}_2$ be arbitrary vectors in $\mathbb{R}^3$. Explain why the vectors are linearly dependent if and only if one of the vectors can be written as a scalar multiple of the other. Then use your ideas to explain why the next theorem is true.

Theorem

Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$. Then the vectors are linearly dependent if and only if one of the vectors can be written as a linear combination of the others vectors.

4. Let $\mathbf{v}_1, \ldots, \mathbf{v}_4$ be arbitrary vectors in $\mathbb{R}^3$. Explain why the vectors must be linearly dependent. Then use your ideas to explain why the next theorem is true.

Theorem

Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$. If $k > n$, then the vectors are linearly dependent. If $k \leq n$, the vectors might be linearly dependent or independent.