07 – Matrix Transformations

Definition: Matrix Transformation

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a **matrix transformation** if it can be written as $T(\mathbf{x}) = A\mathbf{x}$ for some $m \times n$ matrix A.

1. Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, and define $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$.

(a) Compute the image of $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ under T.

(b) Determine if $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$ is in the range of T.

(c) Find a vector $\mathbf{w}$ that is *not* in the range of T.

2. For each transformation below, find the images of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, find the image of an arbitrary vector $\begin{bmatrix} x \\ y \end{bmatrix}$, and try to describe the transformation geometrically.

(a) $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T_1(\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x}$

(b) $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T_2(\mathbf{x}) = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \mathbf{x}$

(c) $T_3 : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T_3(\mathbf{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}$

Theorem

The matrix transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(\mathbf{x}) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \mathbf{x}$$

performs a counter-clockwise rotation of $\mathbb{R}^2$ by an angle of θ.