09 – One-to-one & Onto Transformations

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T.

1. The following are logically equivalent:

- T maps *onto* $\mathbb{R}^m$
- for every $\mathbf{b}$ in $\mathbb{R}^m$, $A\mathbf{x} = \mathbf{b}$ has a solution
- the columns of A span $\mathbb{R}^m$
- A has a pivot position in every row

2. The following are logically equivalent:

- T is *one-to-one*
- for every $\mathbf{b}$ in $\mathbb{R}^m$, $A\mathbf{x} = \mathbf{b}$ has at most one solution
- the columns of A are linearly independent
- A has a pivot position in every column

1. Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 - 2x_3 \\ 4x_1 \\ x_1 - x_2 + x_3 \end{bmatrix}.$$

The standard matrix for T is $\begin{bmatrix} 3 & 0 & -2 \\ 4 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}$. Show that T is both one-to-one and onto $\mathbb{R}^3$.

2. Define $T : \mathbb{R}^4 \to \mathbb{R}^3$ by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 - x_4 \\ 3x_1 - 3x_2 + 4x_3 + 8x_4 \\ 2x_1 - 2x_2 + 2x_3 + 5x_4 \end{bmatrix}.$$

(a) Find the standard matrix for T.

(b) Show that T maps onto $\mathbb{R}^3$ but is *not* one-to-one.

(c) Find two different vectors whose image under T is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

3. Explain why a linear transformation from $\mathbb{R}^7$ to $\mathbb{R}^3$ can *not* be one-to-one.

4. Explain why a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^7$ can *not* map onto $\mathbb{R}^3$.