10 – Matrix Inverses

Definition: Inverse Matrix

Let A be an $n \times n$ matrix. If there exists a matrix B such that $AB = I_n$ and $BA = I_n$, then we say A is **invertible**, and the matrix B is called the **inverse** of A, denoted by A^{-1}.

Theorem

If a matrix is invertible, then there is only one possible inverse of A.

1. Let $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$.

(a) Verify that $A^{-1} = \begin{bmatrix} -3 & 2 \\ \frac{5}{2} & \frac{-3}{2} \end{bmatrix}$ by showing $AA^{-1} = I_2 = A^{-1}A$.

(b) Use the fact that $A^{-1}A = I_2$ to solve $\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

2. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and assume that $ad - bc \neq 0$. Verify that $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Theorem: Formula for A^{-1} when A is 2×2

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Define $\det A = ad - bc$, which is called the **determinant** of A.

1. If $\det A \neq 0$, then $A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

2. If $\det A = 0$, then A is *not* invertible, i.e. A^{-1} does not exist.

Theorem

Let A be an $n \times n$ matrix. Then A is invertible if and only if its RREF is I_n. Further, when A is invertible, any sequence of row operations that transforms A to I_n will also transform I_n to A^{-1}.

Theorem: Algorithm for finding A^{-1} when A is $n \times n$

Let A be an $n \times n$ matrix. Row reduce the augmented matrix $\left[A \mid I_n\right]$ to RREF.

- If the RREF of $\left[A \mid I_n\right]$ is $\left[I_n \mid B\right]$, then A is invertible, and $B = A^{-1}$.
- If the RREF of $\left[A \mid I_n\right]$ is $\left[\textbf{not } I_n \mid B\right]$, then A is not invertible.

3. Find the inverse of A, if it exists.

(a) $\begin{bmatrix} 0 & 3 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 4 & -3 & 7 \end{bmatrix}$