11 – Subspaces & Bases

Definition: Subspace

A subset H of $\mathbb{R}^n$ is called a **subspace** if the following hold:

1. the zero vector $\mathbf{0}$ is in H;

2. if vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are both in H, then $\mathbf{u}_1 + \mathbf{u}_2$ is also in H;

3. if a vector $\mathbf{u}$ is in H and c is a scalar, then $c\mathbf{u}$ is also in H.

1. Let A be an $m \times n$ matrix, and let H be the set of all solutions to $A\mathbf{x} = \mathbf{0}$. Show that H is a subspace of $\mathbb{R}^n$.

2. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be arbitrary vectors in $\mathbb{R}^n$. Show that $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is a subspace of $\mathbb{R}^n$.

Theorem

Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$. Then $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a subspace of $\mathbb{R}^n$.

Definition

Let H be subspace of $\mathbb{R}^n$. Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ form a **basis** for H if

- $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = H$, and
- $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent.

3. Determine which of the following are bases for $\mathbb{R}^3$.

(a) $\begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ -6 \end{bmatrix}$

(b) $\begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ -6 \end{bmatrix}$

(c) $\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ -7 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 5 \end{bmatrix}$

(d) $\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ -7 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$