12 – Null Space and Column Space

Definition: Null Space

The **null space** of a matrix A is the set of *all* solutions to $A\mathbf{x} = \mathbf{0}$. It is denoted $\operatorname{Nul} A$.

1. Let $A = \begin{bmatrix} 1 & 4 & 8 & -3 & -7 \\ -1 & 2 & 7 & 3 & 4 \\ -2 & 2 & 9 & 5 & 5 \\ 3 & 6 & 9 & -5 & -2 \end{bmatrix}$. Determine if $\begin{bmatrix} 4 \\ -5 \\ 2 \\ 0 \\ 0 \end{bmatrix}$ is in $\operatorname{Nul} A$.

Strategy: Basis for $\operatorname{Nul} A$

Let A be any matrix. To find a basis for $\operatorname{Nul} A$, do the following.

- Solve $A\mathbf{x} = \mathbf{0}$ (usually with row reduction).
- Write the solution set in *parametric vector form* (using the process from class).
- The vectors appearing in the parametric vector form are a basis for $\operatorname{Nul} A$.

2. Find a basis for the null space of the matrix in Exercise **1.** You can use the fact that

$$A \sim \begin{bmatrix} 1 & 0 & -2 & 0 & 7 \\ 0 & 2 & 5 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition: Column Space

The **column space** of a matrix A is the set of *all* linear combinations of the columns of A. It is denoted $\operatorname{Col} A$.

Strategy: Basis for $\operatorname{Col} A$

Let A be any matrix. To find a basis for $\operatorname{Col} A$, do the following.

- Row reduce A to REF, and locate the pivots.
- The columns of the *original* matrix A that correspond to the pivots form a basis for $\operatorname{Col} A$.

3. Find a basis for the column space of the matrix in the previous exercise.

Strategy: Basis for $\operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$

To find a basis for $\operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$, create the matrix $A = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_k \end{bmatrix}$, and then find a basis for $\operatorname{Col} A$ as above.

4. Find a basis for the subspace of $\mathbb{R}^3$ spanned by $\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ -3 \end{bmatrix}$.