14 – Eigenvectors & Eigenvalues

Definition: Eigenvectors & Eigenvalues for Linear Transformations

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation.

- A scalar λ is called an **eigenvalue** of T if $T(\mathbf{x}) = \lambda\mathbf{x}$ has a nontrivial solution.
- Each nontrivial solution to $T(\mathbf{x}) = \lambda\mathbf{x}$ is called an **eigenvector** associated to λ.
- The set of all solutions to $T(\mathbf{x}) = \lambda\mathbf{x}$, denoted $E_\lambda(T)$, is the **eigenspace** of T associated to λ.

1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ denote reflection over the y-axis.

(a) Show that $\lambda = 1$ is an eigenvalue of T. Then describe $E_1(T)$.

(b) Show that $\lambda = -1$ is an eigenvalue of T. Then describe $E_{-1}(T)$.

(c) What other numbers could be eigenvalues of T?

Definition: Eigenvectors & Eigenvalues for Matrices

Let A be an $n \times n$ matrix.

- A scalar λ is called an **eigenvalue** of A if $A\mathbf{x} = \lambda\mathbf{x}$ has a nontrivial solution.
- Each nontrivial solution to $A\mathbf{x} = \lambda\mathbf{x}$ is called an **eigenvector** associated to λ.
- The set of all solutions to $A\mathbf{x} = \lambda\mathbf{x}$, denoted $E_\lambda(A)$, is the **eigenspace** of A associated to λ.

2. Let $A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$. Show that $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is not an eigenvector of A but $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is.

Theorem

Let A be an $n \times n$ matrix and λ any scalar.

1. λ is an eigenvalue of $A \iff (A - \lambda I_n)\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

2. $\mathbf{v}$ is an eignvector associated to $\lambda \iff \mathbf{v} \neq \mathbf{0}$ and $(A - \lambda I_n)\mathbf{v} = \mathbf{0}$.

3. $E_\lambda(A) = \text{Nul}(A - \lambda I_n)$.

3. Let $A = \begin{bmatrix} 3 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

(a) Show that $\lambda = 4$ is an eigenvalue of A, and find a basis for $E_4(A)$.

(b) Show that $\lambda = 3$ is not an eigenvalue of A.