15 – Characteristic Polynomial

Definition: Characteristic Polynomial

Let A be an $n \times n$ matrix.

- The polynomial $p(\lambda) = \det(A - \lambda I)$ is called the **characteristic polynomial** of A.
- The equation $\det(A - \lambda I) = 0$ is called the **characteristic equation** of A.

Theorem

Let A be an $n \times n$ matrix. The eigenvalues of A are precisely the roots of the characteristic polynomial (i.e. the solutions to the characteristic equation).

1. Find the characteristic polynomial of each matrix, and use it to determine the eigenvalues.

(a) $A = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$

(b) $B = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$

(c) $C = \begin{bmatrix} 3 & 4 & -3 \\ 0 & -5 & 7 \\ 0 & 0 & 3 \end{bmatrix}$

Theorem

Let A be an $n \times n$ matrix. If A is upper or lower triangular, then the eigenvalues of A are precisely the entries on the main diagonal.

Definition: Similarity

Let A, B be $n \times n$ matrices. We say that A is **similar** to B if there is an invertible matrix P such that $B = P^{-1}AP$.
Note: if A is similar to B, then B is similar to A. This is because if $B = P^{-1}AP$, then $A = Q^{-1}BQ$ for $Q = P^{-1}$.

2. Suppose that A is similar to B with $B = P^{-1}AP$ for some matrix P. Show that $B^5 = P^{-1}A^5P$.

3. Suppose that A is similar to B. Use properties of the determinant, to show that $\det A = \det B$.

Theorem: Similar Matrices have Similar Properties

Assume A and B are similar matrices. Then

1. A and B have the same determinant;

2. A is invertible if and only if B is invertible;

3. A and B have the same characteristic polynomial;

4. A and B have the same eigenvalues (including multiplicity).