# 17 – Markov Chains

**Definition**

- A vector $\mathbf{v}$ is called a **probability vector** if the entries of $\mathbf{v}$ are nonnegative and add up to 1.
- A square matrix $T$ is called a **stochastic matrix** if every column of $A$ is a probability vector.
- If $T$ is a stochastic matrix, then $\mathbf{q}$ is called a **steady-state vector** for $T$ if $q$ is a probability vector and $T\mathbf{q} = \mathbf{q}$.

**1.** Assume $T$ is a $3 \times 3$ stochastic matrix and $\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ is in $E_1(T)$. Find a steady-state vector for $T$.

**Definition**

A **Markov chain** is an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots$ of probability vectors together with a stochastic matrix $T$ such that $T\mathbf{x}_{k-1} = \mathbf{x}_k$ for all $k \geq 1$.

**Theorem**

Let $T$ be a stochastic matrix such that some power of $T$ contains only positive entries. Then $T$ has a unique steady-state vector $\mathbf{q}$, and every Markov chain of the form $T\mathbf{x}_{k-1} = \mathbf{x}_k$ converges to $\mathbf{q}$ as $k \to \infty$, i.e. $\mathbf{x}_\infty = \mathbf{q}$.

**Algorithm: Page Rank Algorithm**

We want to rank the pages in some network of webpages.

**1.** Create the transition matrix $T$. This will be a stochastic matrix.

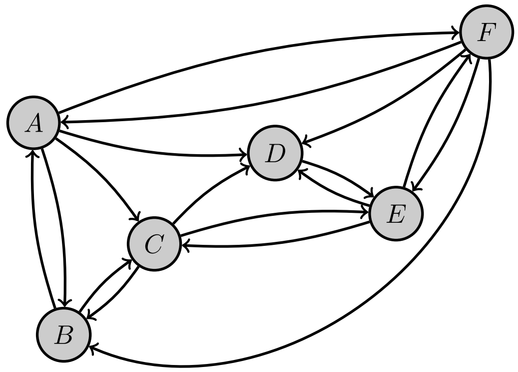
- In our class, we will simply assume that some power of $T$ contains only positive entries. In general, a matrix without this property can be slightly adjusted so that it does.

**2.** Find any eigenvector $\mathbf{v}$ in $E_1(T)$.

**3.** Divide $\mathbf{v}$ by the sum of its entries to get $\mathbf{q}$. This is the steady-state vector for $T$; it gives the probabilities of ending up at each page after infinitely-many random clicks.

**4.** The webpage corresponding to the largest value in $\mathbf{q}$ is ranked first, the page corresponding to the second largest value in $\mathbf{q}$ is ranked second, and so on.

**2.** Consider the following network of webpages below. Each node represents a webpage, and each arrow represents a link from one page to another.




**(a)** Find the transition matrix $T$.

**(b)** Find an eigenvector for $T$ associated to the eigenvalue $\lambda = 1$.
*You can use a computing tool like WolframAlpha.*

**(c)** Use your answer from the previous part to find the steady-state vector for $T$.

**(d)** Find the probabilities of ending up on each page after infinitely-many random clicks.

| | | | | | |
|---|---|---|---|---|---|
| $A$ ________ | $B$ ________ | $C$ ________ | $D$ ________ | $E$ ________ | $F$ ________ |

**(e)** Determine the page ranking. (#1 is the most important.)

| | | | | | |
|---|---|---|---|---|---|
| #1 ______ | #2 ______ | #3 ______ | #4 ______ | #5 ______ | #6 ______ |