18 – Inner Product

Definition: Inner Product

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ be vectors in $\mathbb{R}^n$. The **inner product** (or **dot product**) of $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \cdots + u_n v_n.$$

1. Let $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 7 \\ -2 \end{bmatrix}$. Compute $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{u} \cdot \mathbf{u}$.

Theorem

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

2. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

3. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$

4. $\mathbf{u} \cdot \mathbf{u} \geq 0$ and $\mathbf{u} \cdot \mathbf{u} = 0 \iff \mathbf{u} = 0$

Definition: Length & Distance

- The **length** (or **norm**) of a vector $\mathbf{v}$ is $||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$. We say $\mathbf{v}$ is **unit vector** if $||\mathbf{v}|| = 1$.
- The **distance** between vectors $\mathbf{u}$ and $\mathbf{v}$ is

$$\text{dist}(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}|| = \sqrt{(u_1 - v_1)^2 + \cdots + (u_n - v_n)^2}.$$

2. Let $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$.

(a) Compute $\text{dist}(\mathbf{v}, \mathbf{w})$.

(b) Find a unit vector $\mathbf{u}$ in the same direction as $\mathbf{v}$. Graph $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

Theorem

If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then $\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}|| \cdot ||\mathbf{v}|| \cos\theta$.

Definition: Orthogonality

We say vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if $\mathbf{u} \cdot \mathbf{v} = 0$. Equivalently, $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if the angle between them is $90°$.

3. Let $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$.

(a) Show that $\mathbf{w} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ is not orthogonal to $\mathbf{v}$. What is the angle between $\mathbf{v}$ and $\mathbf{w}$?

(b) Find three different vectors in $\mathbb{R}$ that are orthogonal to $\mathbf{v}$. How many other vectors are orthogonal to $\mathbf{v}$?