19 – Orthogonal Projections

Definition: Projection Onto a Line

Let $L = \text{Span}\{\mathbf{u}\}$ for some nonzero $\mathbf{u}$ in $\mathbb{R}^n$. For any $\mathbf{y}$ in $\mathbb{R}^n$, define the **orthogonal projection of y onto** L to be

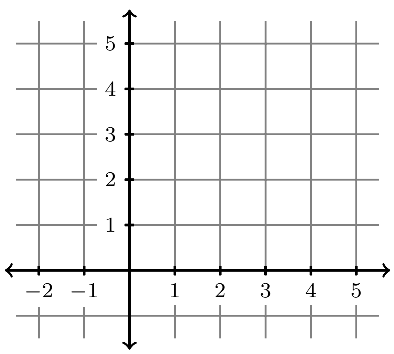
$$\text{proj}_L(\mathbf{y}) = \left(\frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}.$$

Note: we sometimes write $\text{proj}_{\mathbf{u}}(\mathbf{y})$ *in place of* $\text{proj}_L(\mathbf{y})$

1. Let $\mathbf{y} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $L = \text{Span}\left\{\begin{bmatrix} 4 \\ 3 \end{bmatrix}\right\}$.

(a) Compute $\text{proj}_L(\mathbf{y})$.

(b) Let $\hat{\mathbf{y}} = \text{proj}_L(\mathbf{y})$ and $\mathbf{b} = \mathbf{y} - \text{proj}_L(\mathbf{y})$. Graph L, $\mathbf{y}$, $\hat{\mathbf{y}}$, and $\mathbf{b}$.



Definition

A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is said to be **orthogonal** if each pair of vectors in the set is orthogonal. If the set is orthogonal *and* every vector is a unit vector, then the set is said to be **orthonormal**.

2. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1/3 \\ 1/3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 4 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -1 \\ -4 \\ 7 \end{bmatrix}$. Verify that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is orthogonal but *not* orthonormal.

3. Give an example of an orthonormal set of three vectors in $\mathbb{R}^3$.

Theorem

If a set of nonzero vectors forms an orthogonal set, then the vectors are linearly independent.

Definition: Projection Onto a Subspace

Let W be a subspace of $\mathbb{R}^n$, and let $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be any *orthogonal* basis for W. For any $\mathbf{y}$ in $\mathbb{R}^n$, define the **orthogonal projection of $\mathbf{y}$ onto** W to be

$$\text{proj}_W(\mathbf{y}) = \text{proj}_{\mathbf{u}_1}(\mathbf{y}) + \cdots + \text{proj}_{\mathbf{u}_k}(\mathbf{y}) = \left(\frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right)\mathbf{u}_1 + \cdots + \left(\frac{\mathbf{y} \cdot \mathbf{u}_k}{\mathbf{u}_k \cdot \mathbf{u}_k}\right)\mathbf{u}_k.$$

Note: $\text{proj}_W(\mathbf{y})$ *gives the same answer no matter which orthogonal basis you use.*

4. Let $W = \text{Span}\{\mathbf{w}_1, \mathbf{w}_2\}$ for $\mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$.

(a) Verify that $\mathbf{w}_1, \mathbf{w}_2$ is an orthogonal basis for W.

(b) Compute $\text{proj}_W(\mathbf{y})$ for $\mathbf{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$.

(c) Let $\hat{\mathbf{y}} = \text{proj}_W(\mathbf{y})$ and $\mathbf{b} = \mathbf{y} - \text{proj}_W(\mathbf{y})$. Use GeoGebra to graph W, $\mathbf{y}$, $\hat{\mathbf{y}}$, and $\mathbf{b}$.

Theorem: Best Approximation Theorem

Let W be a subspace of $\mathbb{R}^n$, and let $\hat{\mathbf{y}} = \text{proj}_W(\mathbf{y})$. Then $\hat{\mathbf{y}}$ is the vector of W that is *closest* to $\mathbf{y}$ in the sense that $\text{dist}(\mathbf{y}, \hat{\mathbf{y}}) \leq \text{dist}(\mathbf{y}, \mathbf{w})$ for all $\mathbf{w}$ in W.