20 – Least Squares

Definition: Least Squares Solution

Let A be an $m \times n$ matrix, and let $\mathbf{b}$ be in $\mathbb{R}^m$. A vector $\hat{\mathbf{x}}$ is called a **least squares solution** to $A\mathbf{x} = \mathbf{b}$ if $\text{dist}(\mathbf{b}, A\hat{\mathbf{x}}) \leq \text{dist}(\mathbf{b}, A\mathbf{x})$ for all $\mathbf{x}$ in $\mathbb{R}^n$. The number $\text{dist}(\mathbf{b}, A\hat{\mathbf{x}})$ is called the **least squares error**.

1. Show that $\hat{\mathbf{x}}$ is an actual solution to $A\mathbf{x} = \mathbf{b}$ precisely when $\text{dist}(\mathbf{b}, A\hat{\mathbf{x}}) = 0$.

Theorem

Let A be an $m \times n$ matrix, and let $\mathbf{b}$ be in $\mathbb{R}^m$. Then $\hat{\mathbf{x}}$ is a least squares solution to $A\mathbf{x} = \mathbf{b}$ if and only if $\hat{\mathbf{x}}$ is a solution to $A^TA\mathbf{x} = A^T\mathbf{b}$.

2. Consider the system $A\mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix}$.

(a) Show that $A\mathbf{x} = \mathbf{b}$ has no solutions.

(b) Find a least squares solution to $A\mathbf{x} = \mathbf{b}$.

(c) What is the least squares error?

3. Suppose you have the data points: $(0, 2), (-3, 5), (2, 3), (4, 12)$, and you want to model the data using a quadratic function of the form $f(t) = c_0 + c_1 t + c_2 t^2$. Use least squares to find a quadratic function that best fits the data. Then use a graphing tool to plot your answer together with the given data.