04 – Matrix-Vector Products

Definition: Matrix-Vector Product (MVP)

Suppose that A is an $m \times n$ matrix, and let $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ be in $\mathbb{R}^n$. Let $\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_n$ be the columns of A. Then we define the product $A\mathbf{x}$ by

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n.$$

1. Compute the following.

(a) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

(c) $\begin{bmatrix} 7 & -3 \\ 2 & 1 \\ 9 & -6 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ -5 \end{bmatrix}$

Theorem

Suppose that A is an $m \times n$ matrix, and let $\mathbf{b}$ be in $\mathbb{R}^m$. Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ be the columns of A. Then each of the following have the same solution set.

- **Linear system (as a matrix):** $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \mid \mathbf{b} \end{bmatrix}$
- **Matrix equation:** $A\mathbf{x} = \mathbf{b}$
- **Vector equation (using columns of A):** $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$

Theorem

Suppose that A is an $m \times n$ matrix. Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ be the columns of A. Then the following are logically equivalent. (If one is true, they all are; if one is not true, none are.)

(a) A has a pivot position in every row.

(b) For every $\mathbf{b}$ in $\mathbb{R}^m$, the linear system with matrix $\begin{bmatrix} A \mid \mathbf{b} \end{bmatrix}$ has a solution.

(c) For every $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

(d) For every $\mathbf{b}$ in $\mathbb{R}^m$, $\mathbf{b}$ is a linear combination of the columns of A.

(e) The columns of A span $\mathbb{R}^m$: every vector in $\mathbb{R}^m$ is in $\text{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$.

2. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -4 \\ 6 \\ -8 \end{bmatrix}$. Determine if every vector in $\mathbb{R}^3$ is in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$