08 – Linear Transformations

Definition: Linear Transformation

A transformation T is called a **linear transformation** if

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$, and

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$

for all $\mathbf{u}$ and $\mathbf{v}$ in the domain of T and all scalars c.

1. Let A be an arbitrary 4×3 matrix, and let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be defined by $T(\mathbf{x}) = A\mathbf{x}$. Write $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$ (where $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are the columns of A) and use the definition of a matrix-vector product to show that T is a linear transformation.

Theorem

Every matrix transformation is a linear transformation.

2. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation. Assume you know that

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 7 \\ 5 \end{bmatrix}.$$

(a) Find a formula for $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$.

(b) Show that T is a matrix transformation.

Definition: Standard Basis for $\mathbb{R}^n$

We use $\mathbf{e}_k$ to denote the vector with 1 in the k^{th}-entry and 0 in every other entry.
When working in $\mathbb{R}^3$, $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. In $\mathbb{R}^4$, $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ldots$.

Theorem

Every linear transformation is a matrix transformation. Specifically, if $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then $T(\mathbf{x}) = A\mathbf{x}$ where A is the matrix whose j^{th} column is $T(\mathbf{e}_j)$, i.e.

$$A = \begin{bmatrix} T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_j) & \cdots & T(\mathbf{e}_n) \end{bmatrix}$$

The matrix A is called the **standard matrix** of T.

3. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by the rule that T rotates each point counter-clockwise by 60° and then reflects the result over the x-axis. It is a fact that T is a linear transformation; find its standard matrix.