Section 5 — Linear Congruences

Want to solve $ax \equiv b \pmod{m}$.

Q: How many solutions do you expect?
... and what do we mean by "solution"?

Ex Find all solutions to $3x \equiv 6 \pmod{10}$.

Thm 4 - Sect. 4: $(3,10)=1$ so we can cancel.

$3x \equiv 6 \pmod{10} \iff x \equiv 2 \pmod{10}$

Answer 1: $\boxed{x \equiv 2 \pmod{10}}$ 1 solution

Answer 2: $x = 2, 12, 22, \ldots \quad -8, -18, \ldots$

so, $\boxed{x = 2 + 10t \text{ for } t \in \mathbb{Z}.}$ ∞-many sol.

Convention: when we talk about solutions to a congruence (e.g. $ax \equiv b \pmod{m}$) we will only consider least residues mod m.

Ex Find all solutions.

(a) $3x \equiv 6 \pmod{7}$ cancel — 3 is cancellable modulo 7

(b) $3x \equiv 1 \pmod{7}$ $3x \equiv 8 \equiv 15$ ... now cancel

(c) $3x \equiv 1 \pmod{6}$

(d) $3x \equiv 3 \pmod{6}$

} table. (3 is not cancellable modulo 6)

Lemma x_0 is a solution to $ax \equiv b \pmod m$ **iff** for some $y_0 \in \mathbb{Z}$, x_0, y_0 is a solution to $ax + my = b$.

pf

$$ax_0 \equiv b \pmod m \iff m \mid (ax_0 - b)$$
$$\iff ax_0 - b = km \quad \text{for some } k \in \mathbb{Z}$$
$$\iff ax_0 + m(-k) = b \quad (\text{"}y_0\text{"} = -k)$$
$$\iff x_0, -k \text{ is a solution to } ax + my = b$$

□

Find all (least residue) solutions to $ax \equiv b \pmod m$ $\longleftrightarrow$ Find all x-values of the solutions to $ax + my = b$ with $0 \le x < m$.

Theorem 1 Consider $ax \equiv b \pmod m$. Let $d = (a, m)$.

1. If $d \nmid b$, then there are no solutions.
2. If $d \mid b$, then there are exactly d solutions.

↑ counting only least residues.

pf idea

solutions to $ax \equiv b \pmod m$ $\longleftrightarrow$ x-values of solutions to $ax + my = b$, $0 \le x < m$

1. If $d \nmid b$, there are no solutions by Lemma 2 – Sec. 3.

Optional

2. Assume $d \mid b$. Note that $d \mid a$, $d \mid m$, $d \mid b$ so $ax + my = b$ is not reduced.

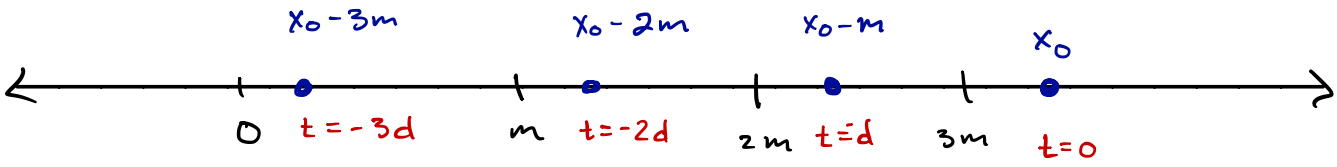
However,

$$\frac{a}{d}x + \frac{m}{d}y = \frac{b}{d}$$

is reduced. Let x_0, y_0 be one sol. Then all solutions are given by

$$x = x_0 + \frac{m}{d}t \qquad t \in \mathbb{Z}.$$
$$y = y_0 - \frac{a}{d}t$$

we want solutions with $0 \le x < m$.



Thus for some value of t, there is a solution with $0 \le x < m$. Let x_1, y_1 be a solution with x_1 positive and as small as possible. Rewrite the solutions to $\frac{a}{d}x + \frac{m}{d}y = \frac{b}{d}$ as

$$x = x_1 + \frac{m}{d}s \qquad s \in \mathbb{Z}.$$
$$y = y_1 - \frac{a}{d}s$$

Then, $0 \le x_1, x_1 + \frac{m}{d}, x_1 + \frac{m}{d}\cdot 2, \ldots, x_1 \frac{m}{d}\cdot(d-1) < m$

and these are exactly the solutions to $ax \equiv b \pmod{m}$. □

optional

We now know exactly how many solutions to expect — How do we find them?

Ex Solve each of the following.

① $\overset{a}{4}x \equiv \overset{b}{2} \pmod{\overset{m}{12}}$

$(a,m) = (4,12) = 4$. $4 \nmid 2 \Rightarrow$ **No solutions**

② $5x \equiv 2 \pmod{12}$

$(5,12) = 1 \Rightarrow$ 1 solution.

option 1

guess/check/table

x	0	1	2	3	4	5	6	7	8	9	10
5x	0	5	10	15	20	25	30	35	40	45	50
				⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
				3	8	1	6	11	4	9	2

$x \equiv 10 \pmod{12}$

option 2

manipulate then cancel

$5x \equiv 2 \Rightarrow 5x \equiv 2 + 12 + 12 + 12 + \cdots \pmod{12}$

$\Rightarrow 5x \equiv 2 + 48 \pmod{12}$

$\Rightarrow 5x \equiv 50 \pmod{12}$

5 is cancellable mod 12, b/c $(5,12) = 1$

$\Rightarrow \boxed{x \equiv 10} \pmod{12}$

③ $9x \equiv 6 \pmod{12}$

$(9,12) = 3$, $3 \mid 6 \Rightarrow$ 3 solutions

Option 1

table

x	0	1	2	3	4	5	6	7	8	9	10	11
9x	0	9	18	27	36	45	54	63	72	9	6	3
			‖	‖	‖	‖	‖	‖	‖			
			6	3	0	7	6	3	0			

repeat repeat

$x \equiv 2, 6, 10$

Option 2

* manipulate and cancel

$9x \equiv 6 \pmod{12}$ ← $(9,12) \neq 1$ so can't cancel ∴

* use Thm 5 – Sect. 4

(a) Solve the reduced congruence

$3x \equiv 2 \pmod 4$

$3x \equiv 2 \pmod 4 \iff 3x \equiv 6 \pmod 4$

$\iff x \equiv 2 \pmod 4$

(b) Solve the original

- we found

$9x \equiv 6 \pmod{12} \Rightarrow x \equiv 2 \pmod 4$

- Want all residues mod 12 cong. to 2 mod 4

$2 \equiv 2 \mod 4$

$6 \equiv 2 \mod 4$

$10 \equiv 2 \mod 4$

Another Example...

Ex Solve $12x \equiv 20 \pmod{28}$

① Find # of sol. : $(12, 28) = 4$ AND $4 \mid 20$ so 4 solutions

② Solve

$12x \equiv 20 \pmod{28}$ —reduce→ $3x \equiv 5 \pmod{7}$

3 is cancellable mod 7

$3x \equiv 12 \pmod{7}$

$x \equiv 4 \pmod{7}$ ← back to 28

want ALL $x \pmod{28}$ such that $x \equiv 4 \pmod{7}$

$4 \equiv 4+7 = 11$

$\equiv 4+14 = 18$

$\equiv 4+21 = 25$

ANSWER: $x \equiv 4, 11, 18, 25 \pmod{28}$

Systems of congruences

Ex Solve $x+2y \equiv 4 \pmod{11}$, $3x+y \equiv -1 \pmod{11}$

add $-2 \cdot (3x + y \equiv -1)$

$x+2y \equiv 4 \pmod{11}$

$-6x - 2y \equiv 2$

$-5x \equiv 6 \pmod{11}$

$-5x \equiv 6 \equiv -5 \Rightarrow$ cancellable $\Rightarrow x \equiv 1$

$1 + 2y \equiv_{11} 4$

$2y \equiv_{11} 3$

$2y \equiv_{11} 14$

$y \equiv 7$

Systems of congruences with different moduli

Ever notice that 48, 49, 50 are each divisible by a square... (16, 49, 25) here's a related problem...

Ex Find n such that $3^2 \mid n,\ 4^2 \mid n+1,\ 5^2 \mid n+2$

Want to solve

① $n \equiv 0 \mod 9$
② $n+1 \equiv 0 \mod 16$
③ $n+2 \equiv 0 \mod 25$

} system of cong.

① $n \equiv 0 \mod 9 \Rightarrow \boxed{n = 9r}$ for $r \in \mathbb{Z}$

9 (and 3) are cancellable mod 16

② $n+1 \equiv 0 \mod 16 \Rightarrow 9r+1 \equiv 0 \pmod{16}$

$\Rightarrow 9r \equiv -1 \pmod{16}$

$\equiv 15 \equiv 31 \equiv 47 \equiv 63$

$\Rightarrow r \equiv 7 \pmod{16}$

$\Rightarrow r = 7 + 16s$

$\Rightarrow n = 9(7+16s)$

$\boxed{n = 63 + 144s}$

③ $n+2 \equiv 0 \pmod{25} \Rightarrow 65 + 144s \equiv 0 \pmod{25}$

$\Rightarrow 15 + 19s \equiv 0$

$19s \equiv -15$

$-6s \equiv 10 \equiv 35 \equiv 60$

$s \equiv -10 \equiv 15 \pmod{25}$

$\Rightarrow s = 15 + 25t$

$n = 63 + 144(15+25t)$

$\boxed{n = 2223 + 3600t}$

Any t works — $t = 0$

$\boxed{2223,\ 2224,\ 2225}$

$3^2 \cdot 13 \cdot 19,\ 4^2 \cdot 139,\ 5^2 \cdot 89$

$\Leftrightarrow n \equiv 2223 \pmod{9 \cdot 16 \cdot 25}$

product of moduli

This process often works...

Thm 2 (Chinese Remainder Theorem)

Consider the system of congruences:

$$\begin{aligned} n &\equiv a_1 \pmod{m_1} \\ n &\equiv a_2 \pmod{m_2} \\ &\vdots \\ n &\equiv a_k \pmod{m_k} \end{aligned}$$

If $(m_i, m_j) = 1$ for all $i \neq j$, then the system has a unique solution modulo $m_1 \cdot m_2 \cdot \cdots \cdot m_k$.

Even if the moduli are not pairwise relatively prime, the system may still have solutions.

Ex Find the smallest odd n, $n > 3$, s.t. $3 \mid n$, $5 \mid n+2$, and $7 \mid n+4$.

Need to solve the system:

$$\begin{aligned} n &\equiv 0 \pmod 3 \\ n+2 &\equiv 0 \pmod 5 \\ n+4 &\equiv 0 \pmod 7 \\ n &\equiv 1 \pmod 2 \end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned} n &\equiv 0 \pmod 3 \\ n &\equiv -2 \pmod 5 \\ n &\equiv -4 \pmod 7 \\ n &\equiv 1 \pmod 2 \end{aligned}$$

moduli are rel. prime so we are guaranteed to have a sol. — now we need to find it — HW!