

# Section 7 - Divisors of an Integer

Def Let  $n \in \mathbb{Z}^+$ .

- $d(n)$  is the number of positive divisors of  $n$ .
- $\sigma(n)$  is the sum of the positive divisors of  $n$ .

Ex Find  $d(6), d(7), d(8)$ . what about  $\sigma(6), \sigma(7), \sigma(8)$ .

Ex Fill in the table

$n$	1	2	3	4	5	6	7	8	9	10
$d(n)$	1	2	2	3	2	4	2	4	3	4
$\sigma(n)$	1	3	4	7	6	12	8	15	13	18

Ex Find  $d(5), d(5^2), d(5^3)$

Lemma If  $p$  is prime and  $n \in \mathbb{Z}^+$ , then  $d(p^n) = \underline{n+1}$

pf divisors of  $p^n$  are  $1, p^1, p^2, \dots, p^n$

Lemma If  $p$  is prime and  $n \in \mathbb{Z}^+$ , then  $\sigma(p^n) = \underline{??}$

Hint:  $\sigma(p^n) = 1 + p + p^2 + \dots + p^n$  is a finite geo. series.

Ex Suppose  $p, q$  are different primes and  $m = p^2 q^3$   
compare  $d(p^2), d(q^3), d(m)$ .

	1	$q$	$q^2$	$q^3$
1	1	$2$	$2^2$	$2^3$
$p$	$p$	$p^2$	$p^2 q$	$p^2 q^2$
$p^2$	$p^2$	$p^2 q$	$p^2 q^2$	$p^2 q^3$

← could there be repeats?

No by FTA

Theorem 1 If  $n \in \mathbb{Z}^+$ , and  $n = p_1^{e_1} \cdots p_k^{e_k}$  is its prime-power decomposition, then

$$d(n) = d(p_1^{e_1}) \cdots d(p_k^{e_k}).$$

$$\text{Thus, } d(n) = (e_1 + 1) \cdots (e_k + 1).$$

Prf

Consider the numbers of the form  $p_1^{f_1} \cdots p_k^{f_k}$  where  $0 \leq f_i \leq e_i$ . Every divisor of  $n$  must be of this form by FTA, and every number of this form is a divisor. Thus the divisors of  $n$  are exactly the numbers of this form. How many such numbers are there? ...  $(e_1 + 1)$  choices for  $f_1$ ,  $(e_2 + 1)$  choices for  $f_2$ , ...,  $(e_k + 1)$  choices for  $f_k$ .

$$\text{Thus, } d(n) = (e_1 + 1) \cdots (e_k + 1).$$

$$\stackrel{\parallel}{d(p_1^{e_1})} \cdots \stackrel{\parallel}{d(p_k^{e_k})}.$$

← by previous lemma.

□

Ex Compute  $d$  for  $46,585,000 = 2^{e_1} \cdot 5^{e_2} \cdot 7^{e_3} \cdot 11^{e_4}$

$$\begin{aligned} d(46585000) &= d(2^3) d(5^4) d(7^1) d(11^3) \\ &= 4 \cdot 5 \cdot 2 \cdot 4 = \boxed{160} \end{aligned}$$

Def Let  $f$  be a function from  $\mathbb{Z}^+$  to  $\mathbb{Z}^+$ .

We say  $f$  is multiplicative if whenever  $(m, n) = 1$ ,

$$f(mn) = f(m)f(n).$$

Theorem 3 The function  $\sigma$  is multiplicative.

pt. see book.

\* This is nice to know but Thm 1 is more helpful for us.

Ex Compute  $\sigma(2^4)$ .

$$\text{Div. of } 2^4: 1, 2, 2^2, 2^3, 2^4 \Rightarrow \sigma(2^4) = 1 + 2 + 2^2 + 2^3 + 2^4 \\ = 1 + 2 + 4 + 8 + 16 = \boxed{31}$$

Lemma If  $p$  is prime and  $n \in \mathbb{Z}^+$ , then  $\sigma(p^n) = \frac{p^{n+1} - 1}{p - 1}$

pt  $\sigma(p^n) = 1 + p + p^2 + \dots + p^n$ . Note that

$$(1 + x + x^2 + \dots + x^n) \cdot (1 - x) = (1 + x + x^2 + \dots + x^n) - (x + x^2 + x^3 + \dots + x^{n+1}) \\ = 1 - x^{n+1}$$

Thus,

$$1 + x + x^2 + \dots + x^n = \frac{1 - x^{n+1}}{1 - x} \quad (\text{if } x \neq 1),$$

so

$$\sigma(p^n) = \frac{1 - p^{n+1}}{1 - p} = \frac{p^{n+1} - 1}{p - 1}$$

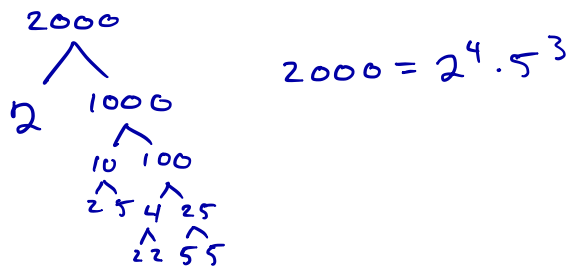
□

Ex Compute  $\sigma(2^4)$  and  $\sigma(5^3)$ .

$$\sigma(2^4) = \frac{2^5 - 1}{2 - 1} = \frac{32 - 1}{1} = \boxed{31}$$

$$\sigma(5^3) = \frac{5^4 - 1}{5 - 1} = \frac{624}{4} = \boxed{156}$$

Ex Compute  $\sigma$  of 2000.



Divisors are  $2^a \cdot 5^b$   $a = 0, 1, 2, 3, 4; b = 0, 1, 2, 3$

		a					
		0	1	2	3	4	
b	0	1	$2^1$	$2^2$	$2^3$	$2^4$	Sum $\rightarrow (1+2^1+2^2+2^3+2^4)$
	1	5	$2^1 \cdot 5$	$2^2 \cdot 5$	$2^3 \cdot 5$	$2^4 \cdot 5$	$\rightarrow (1+2^1+2^2+2^3+2^4) \cdot 5$
	2	$5^2$	$2^1 \cdot 5^2$	$2^2 \cdot 5^2$	$2^3 \cdot 5^2$	$2^4 \cdot 5^2$	$\rightarrow (1+2^1+2^2+2^3+2^4) \cdot 5^2$
	3	$5^3$	$2^1 \cdot 5^3$	$2^2 \cdot 5^3$	$2^3 \cdot 5^3$	$2^4 \cdot 5^3$	$\rightarrow (1+2^1+2^2+2^3+2^4) \cdot 5^3$
							+ <hr style="border: 1px solid green;"/> $(1+2^1+2^2+2^3+2^4) \cdot (1+5+5^2+5^3)$

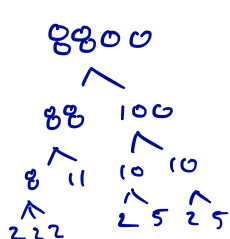
Sum  $\downarrow$

$$\sigma(2000) = \sigma(2^5) \sigma(5^4) = 31 \cdot 156 = \boxed{4836}$$

Theorem 2 If  $n \in \mathbb{Z}^+$ , and  $n = p_1^{e_1} \cdots p_k^{e_k}$  is its prime-power decomposition, then

$$\sigma(n) = \sigma(p_1^{e_1}) \cdots \sigma(p_k^{e_k})$$

Ex Compute  $\sigma$  of 8800



$$8800 = 2^5 \cdot 5^2 \cdot 11$$

$$\sigma(8800) = \sigma(2^5) \cdot \sigma(5^2) \cdot \sigma(11)$$

$\nearrow \frac{2^6-1}{2-1}$      $\nearrow 1+5+5^2$      $\nearrow 1+11$

$$= 63 \cdot 31 \cdot 12 = \boxed{23436}$$

Theorem 4 The function  $\sigma$  is multiplicative.

Pr. see book.