Course Introduction

What is this course about? ... the integers.

Notation

- Integers: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
- Positive integers: $\mathbb{Z}_{>0} = \{1, 2, 3, \ldots\}$
- Nonnegative integers: $\mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \ldots\}$
- Rational numbers: $\mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z} \text{ and } b \neq 0 \right\}$ is an element of, such that
- Real numbers: $\mathbb{R}$ ← includes $\frac{1}{2}, -\sqrt{2}, \pi, \ldots$

Familiar Problem Find all real solutions

(a) $2x = 1$ $\qquad x = 1/2$

(b) $x(x^2 - 2) = 0$ $\qquad x = 0, \pm\sqrt{2}$

(c) $x^2 = -1$ $\qquad$ no solution (in $\mathbb{R}$)

(d) $x^2 + y^2 = 25$ $\qquad$ all points on circle of radius 5 centered at the origin

Number-theoretic (Diophantine) Problem Find all integer solutions

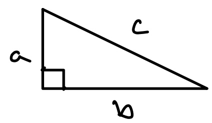
(a) $2x = 1$ $\qquad$ no sol. (in $\mathbb{Z}$)

(b) $x(x^2 - 2) = 0$ $\qquad x = 0$

(c) $x^2 = -1$ $\qquad$ no sol. (in $\mathbb{Z}$)

(d) $x^2 + y^2 = 25$ $\qquad$ Hmm... try a table, e.g. $(\pm 0, \pm 5)$, $(\pm 3, \pm 4)$ in either order

Another Diophantine Problem

Recall: 

$\xleftrightarrow{\text{Pythag. thm}}$ $a^2+b^2=c^2$

law of cosines (via pythag. thm)

- Pythagoras of Samos (Greek, 6th century BCE)
  - ↳ studied in Egypt and result may have bee known to Babylonians ($\sim$ 1900–540 BCE)
- Result was independently discovered by Mesopotamians, Indians, and Chinese (in the least)

Q: what right triangles do you know with integer sides?

$(3,4,5) \to (6,8,10) \to (12,16,20) \to \cdots$

OR

$(0,1,1) \to (0,2,2) \to \cdots$

Problem: Find all integer solutions to $x^2+y^2=z^2$.

↳ called Pythagorean triples

- This is a problem we will probably look at later in the course.

More Problems

so ignoring solutions like $(0,1,1)$

- $x^2+y^2=z^2$ has ∞-many solutions in $\mathbb{Z}_{>0}$
- what about $x^3+y^3=z^3$?
- what about $x^4+y^4=z^4$?

<u>Fermat's Last Theorem</u> If $n \geq 3$, then

$x^n + y^n = z^n$ has <u>no</u> positive integer solutions.

- published in 1995 by Andrew Wiles (Oxford) and resulted in big prizes

Here's another problem...

What are the prime numbers? 2, 3, 5, 7, 11, ...

Did you ever notice that:

$4 = 2+2$ $\quad$ $12 = 5+7$
$6 = 3+3$ $\quad$ $14 = 7+7$
$8 = 3+5$ $\quad$ $16 =$ ← do you see the game we are playing? Can you continue?
$10 = 5+5$

<u>Goldbach's Conjecture</u> Every <u>even</u> integer greater than 2 can be expressed as the sum of two primes.

- unproven (known to be true up to 10^{17})
- the conjecture implies that every integer greater than 5 can be written as the sum of 3 primes
- <u>Theorem (Helfgott - 2013)</u> Every <u>odd</u> integer greater than 5 can be expressed as the sum of 3 primes