Section 3 - Linear Diophatine Eqs.

WS-02 Intro Linear Dioph. (see next page)

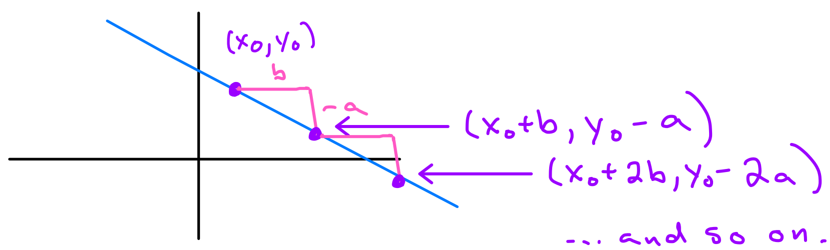
Lingering question: were there other solutions

Important take-away: If there is one integer solution, then there are infinitely many (ignoring additional restrictions).

solutions

So, one integer solution (x_0, y_0) to $ax+by=c$ yielded ∞-many integer solutions. How exactly?

$$ax+by=c \longleftrightarrow y = \frac{-a}{b} + c \quad (\text{if } b \neq 0)$$



... and so on... in both directions.

omitted in book, but it is always implied!!

Lemma 1 Let $a,b,c \in \mathbb{Z}$. If (x_0, y_0) is an integer solution of $ax+by=c$, then so is

$x = x_0 + bt$
$y = y_0 - at$ $\longrightarrow (x_0+bt, y_0-at)$ for all $t \in \mathbb{Z}$.

02 – Introduction to linear Diophantine equations

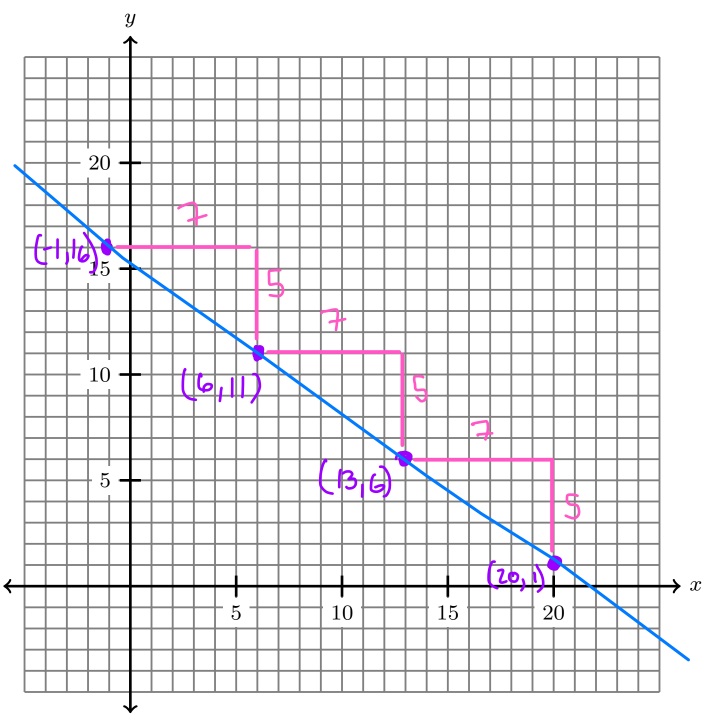
It's my first day selling burgers and burritos on the beach. At end of my shift, I sold \$107 of food. Then I remember that my boss wanted me to keep track of how many of each I sold. All I remember is that I sold more burritos than burgers. Can you help me? Also, burgers are \$5 and burritos are \$7.

1. Let x denote the number of burgers sold and y the number of burritos sold. Based on the story, what equation do you want to solve? What are the restrictions on the variables?

$\boxed{5x + 7y = 107}$ $\quad x, y \in \mathbb{Z}$

$0 \leq x \leq y$

2. Graph the equation you found in the previous part.



$5x + 7y = 107$

$y = -\frac{5}{7}x + \frac{107}{7}$

key observation: (20,1) is on the graph

point: (20,1)

slope: $-5/7$

3. First, *ignore* the fact that I sold more burritos than burgers. Find as many integer solutions as possible to your equation.

(20,1), (13,6), (6,11), (-1,16), ... infinitely-many more if variables can be negative

4. Now, *make use* of the fact that I sold more burritos than burgers to determine how many burritos and burgers I sold? Why? Could there possibly be a different answer?

burgers: $x = 6$

burritos: $y = 11$

Between these integer solutions there do not seem to be other integer solutions, so (6,11) seems to be the only one with $0 \leq x < y$.

__pf__

We are assuming $ax_0 + by_0 = c$. Observe that

$$
\begin{aligned}
a(x_0 + bt) + b(y_0 - at) &= ax_0 + \cancel{abt} + bx_0 - \cancel{bat} \\
&= ax_0 + bx_0 \\
&= c
\end{aligned}
$$

So, $x = x_0 + bt$, $y = y_0 - at$ is also a solution. $\square$

__Ex__ Find ∞-many integer solutions to $5x + 7y = 107$, if possible.

trial and error for now

① Find 1 solution: $x = 20$, $y = 1$ is a solution since $5 \cdot 20 + 7 \cdot 1 = 107$

② use Lemm 1: $\boxed{x = 20 + 7t,\ y = 1 - 5t \quad \text{for } t \in \mathbb{Z}.}$

For example: $\ldots, \underset{t=-1}{(13, 6)}, \underset{t=0}{(20, 1)}, \underset{t=1}{(27, -4)}, \underset{t=2}{(34, -9)}, \ldots$

__Ex__ Find ∞-many integer solutions to $6x + 8y = 107$, if possible.

$\boxed{\text{No solutions.}}$ $2 \mid \text{LHS}$ but $2 \nmid \text{RHS}$

__Ex__ Find ∞-many integer solutions to $6x + 9y = 107$, if possible.

$\boxed{\text{No solutions.}}$ $3 \mid \text{LHS}$ but $3 \nmid \text{RHS}$

Lemma 2 Let $a, b, c \in \mathbb{Z}$. Then

it has ∞-many integer solutions $\iff$ $ax + by = c$ has an integer solution $\iff$ $\gcd(a,b) \mid c$

pf

Let $d = \gcd(a,b)$.

① Assume $ax+by=c$ has an integer solution (x_0, y_0).
Then $ax_0 + by_0 = c$. Since $d \mid a$ and $d \mid b$, $d \mid c$.

② Assume $d \mid c$. Then $c = md$ for some $m \in \mathbb{Z}$.

Theorem 4 section 1 $\implies$ $ax_0 + by_0 = d$ for some $x_0, y_0 \in \mathbb{Z}$

$\implies$ $a(mx_0) + b(my_0) = md = c$

$\implies$ $x = mx_0$, $y = my_0$ is an integer solution to $ax+by=c$ $\square$

Summary so far about $ax+by=c$:

it has ∞-many integer solutions $\iff$ it has one integer solution $\iff$ $\gcd(a,b) \mid c$

But, we have <u>not</u> determined how to find <u>all</u> solutions.

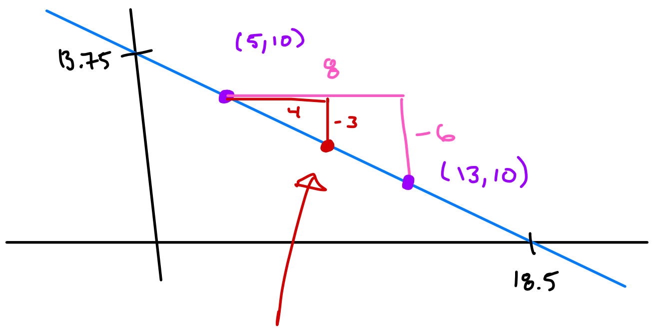
If (x_0, y_0) is a solution to $6x + 8y = 110$, is it true that every solution is of the form

think $(5,10)$

$$x = x_0 + 8t$$
$$y = y_0 - 6t$$
?

what is the slope?

$$-\frac{6}{8} = -\frac{3}{4}$$



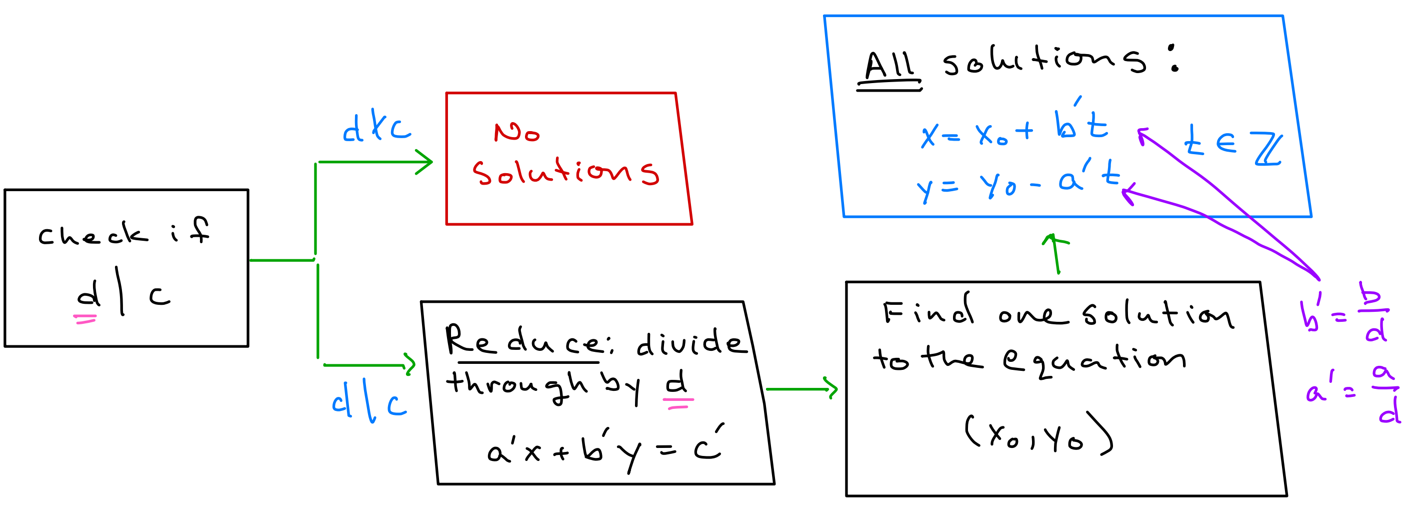
(9,7) we missed this one!

! need to reduce the equation first:

$$6x + 8y = 110 \longrightarrow \boxed{3x + 4y = 55}$$

use this one

Strategy Let $a, b, c \in \mathbb{Z}$. Suppose you want all integer solutions to $ax + by = c$. Let $d = \gcd(a,b)$.



$$a' = \frac{a}{d}, \quad b' = \frac{b}{d}, \quad c' = \frac{c}{d}$$

But how do we know that we get <u>all</u> solutions?

<u>Lemma 3</u> Let $a, b, c \in \mathbb{Z}$. Let (x_0, y_0) be any integer solution to $ax+by=c$. If $\gcd(a,b)=1$, then all integer solutions are given by

$$x = x_0 + bt \qquad t \in \mathbb{Z}.$$
$$y = y_0 - at$$

this means the equation is reduced.

Optional

<u>pf</u> Let (r, s) be any integer solution. Then

$$\begin{array}{rl} & ar+bs = c \\ - & \underline{ax_0+by_0 = c} \end{array}$$

$$\Longrightarrow \quad ar - ax_0 + bs - by_0 = 0$$

$$\Longrightarrow \quad a(r-x_0) + b(s-y_0) = 0$$

$$\Longrightarrow \quad a(r-x_0) = -b(s-y_0)$$

$$\Longrightarrow \quad a(r-x_0) = b(y_0 - s)$$

$$\Longrightarrow \quad a \mid b(y_0 - s) \xRightarrow[(a,b)=1]{\text{cor. 1, sect. 1}} a \mid (y_0 - s)$$

$$\Longrightarrow \quad \boxed{y_0 - s = a \cdot t} \quad \text{for some } t \in \mathbb{Z}$$

$$\Longrightarrow \quad a(r-x_0) = b a \cdot t$$

$$\Longrightarrow \quad \boxed{r - x_0 = bt}$$

$$\Longrightarrow \quad \begin{array}{l} r = x_0 + bt \\ s = y_0 - at \end{array} \quad \text{for } t \in \mathbb{Z}$$

$\square$

Ex Find all integer solutions to $-8x + 20y = 148$.

① compute $\gcd(a,b)$.

$$\gcd(-8, 20) = \boxed{4}$$

② Does $\gcd(a,b)$ divide c?

Yes! $148 = 4 \cdot 37$

So there are solutions — keep going.

③ Reduce: divide through by $\gcd(a,b)$.

divide through by 4

$$\boxed{-2x + 5y = 37}$$ ← reduced equation

④ Find one solution

Hmmm... $x = -1$, $y = 7$ (guess & check)

⑤ write down all solutions (using reduced eq.)

$$x = x_0 + b't$$
$$y = y_0 - a't \qquad t \in \mathbb{Z}$$

from reduced equation

so

$$\boxed{\begin{aligned} x &= -1 + 5t \\ y &= 7 + 2t \end{aligned} \qquad t \in \mathbb{Z}}$$

<u>Warm-up</u> Revisit Burger & Burrito problem but now burgers are \$3 and burritos are \$7. I sold \$108 in total and I know I sold an even number of burgers. What are the possibilities for the number of each I sold (see WS-02)

Hint: 108 is divisible by 3.

Let x = # burgers sold
y = # burritos sold

$$\boxed{3x + 7y = 108 \qquad x, y \in \mathbb{Z}}$$

and x is even, $x, y \geq 0$

1. $\gcd(3,7) = 1$
2. $1 \mid 108$ so there are solutions
3. equation is already reduced
4. Find one solution:

$$108 = 3 \cdot 36 \Rightarrow 3 \cdot 36 + 7 \cdot 0 = 108$$

$\Rightarrow$ $x = 36$, $y = 0$ is a solution

5. Find all solutions

$$\begin{aligned} x &= x_0 + bt \\ y &= y_0 - at \end{aligned} \Rightarrow \begin{aligned} x &= 36 + 7t \\ y &= 0 - 3t \end{aligned} \qquad t \in \mathbb{Z}$$

6. Answer the question

t	1	0	-1	-2	-3	-4	-5	-6	...
(x,y)	~~(43,-3)~~	(36,0)	~~(29,3)~~	(22,6)	~~(15,9)~~	(8,12)	~~(1,15)~~	~~(-6,18)~~	

$$\boxed{(36,0),\ (22,6),\ (8,12)}$$