Section 7 — Divisors of an Integer

Def Let $n \in \mathbb{Z}^+$.

- $d(n)$ is the number of positive divisors of n.
- $\sigma(n)$ is the sum of the positive divisors of n.

Ex Find $d(6), d(7), d(8)$. what about $\sigma(6), \sigma(7), \sigma(8)$.

Ex Fill in the table

n	1	2	3	4	5	6	7	8	9	10
$d(n)$	1	2	2	3	2	4	2	4	3	4
$\sigma(n)$	1	3	4	7	6	12	8	15	13	18

Ex Find $d(5), d(5^2), d(5^3)$

Lemma If p is prime and $n \in \mathbb{Z}^+$, then $d(p^n) = \underline{n+1}$

pf divisors of p^n are $1, p^1, p^2, \ldots, p^n$

Lemma If p is prime and $n \in \mathbb{Z}^+$, then $\sigma(p^n) = \underline{\;??\;}$

Hint: $\sigma(p^n) = 1 + p + p^2 + \cdots + p^n$ is a finite geo. series.

Ex Suppose p, q are different primes and $m = p^2 q^3$
Compare $d(p^2), d(q^3), d(m)$.

	1	q	q^2	q^3
1	1	q	q^2	q^3
p	p	pq	pq^2	pq^3
p^2	p^2	p^2q	p^2q^2	p^2q^3

could there be repeats?
No by FTA

Theorem 1 If $n \in \mathbb{Z}^+$, and $n = p_1^{e_1} \cdots p_k^{e_k}$ is its prime-power decomposition, then

$$d(n) = d(p_1^{e_1}) \cdots d(p_k^{e_k}).$$

Thus, $d(n) = (e_1+1) \cdots (e_k+1)$.

Pf

Consider the numbers of the form $p_1^{f_1} \cdots p_k^{f_k}$ where $0 \le f_i \le e_i$. Every divisor of n must be of this form by FTA, and every number of this form is a divisor. Thus the divisors of n are exactly the numbers of this form. How many such numbers are there? ... (e_1+1) choices for f_1, (e_2+1) choices for f_2, ..., (e_k+1) choices for f_k.

Thus, $d(n) = \underset{d(p_1^{e_1})}{\underset{\shortparallel}{(e_1+1)}} \cdots \underset{d(p_k^{e_k})}{\underset{\shortparallel}{(e_k+1)}}$. $\leftarrow$ by previous lemma.

$\square$

Ex Compute d for $46{,}585{,}000 = 2^{\overset{e_1}{3}} \cdot 5^{\overset{e_2}{4}} \cdot 7^{\overset{e_3}{1}} \cdot 11^{\overset{e_4}{3}}$

$$d(46585000) = d(2^3)\,d(5^4)\,d(7^1)\,d(11^3)$$
$$= 4 \cdot 5 \cdot 2 \cdot 4 = \boxed{160}$$

Def Let f be a function from $\mathbb{Z}^+$ to $\mathbb{Z}^+$. We say f is <u>multiplicative</u> if whenever $(m,n)=1$, $f(mn) = f(m)f(n)$.

Theorem 3 The function d is multiplicative.

pf. see book.

* In general, Theorem 1 is more helpful with computations — Theorem 3 can be helpful with theoretical applications

Ex Let $m \in \mathbb{Z}_{>0}$. If m is odd, show that $d(2m)$ is even.

$\gcd(2,m)=1 \implies d(2m)=d(2)d(m)$
$\implies d(2m)=2\cdot d(m)$ (since 2 is prime)
$\implies d(2m)$ is even.

* Also, if $p \nmid m$ then $d(pm)$ is again even.

Back to σ...

Ex Compute $\sigma(2^4)$.

Div. of 2^4 : $1,2,2^2,2^3,2^4 \implies \sigma(2^4)=1+2+2^2+2^3+2^4=1+2+4+8+16=\boxed{31}$

Lemma If p is prime and $n \in \mathbb{Z}^+$, then $\sigma(p^n)=\dfrac{p^{n+1}-1}{p-1}$

pf. $\sigma(p^n)=1+p+p^2+\cdots+p^n$. Note that

$$(1+x+x^2+\cdots+x^n)\cdot(1-x)=(1+x+x^2+\cdots+x^n)-(x+x^2+x^3+\cdots+x^{n+1})$$
$$=1-x^{n+1}$$

Thus,

$$1+x+x^2+\cdots+x^n=\frac{1-x^{n+1}}{1-x} \quad (\text{if } x\neq 1),$$

so

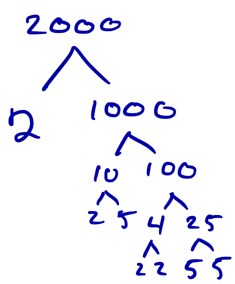
$$\sigma(p^n)=\frac{1-p^{n+1}}{1-p}=\frac{p^{n+1}-1}{p-1} \quad \square$$

Ex compute $\sigma(2^4)$ and $\sigma(5^3)$.

$$\sigma(2^4) = \frac{2^5-1}{2-1} = \frac{32-1}{1} = \boxed{31}$$

$$\sigma(5^3) = \frac{5^4-1}{5-1} = \frac{624}{4} = \boxed{156}$$

Ex Compute σ of 2000.



$2000 = 2^4 \cdot 5^3$

Divisors are $2^a \cdot 5^b$ $\quad a = 0,1,2,3,4;\ b = 0,1,2,3$

b \ a	0	1	2	3	4
0	1	2^1	2^2	2^3	2^4
1	5	$2^1\cdot 5$	$2^2\cdot 5$	$2^3\cdot 5$	$2^4\cdot 5$
2	5^2	$2^1\cdot 5^2$	$2^2 5^2$	$2^3 5^2$	$2^4\cdot 5^3$
3	5^3	$2^1 5^3$	$2^2\cdot 5^3$	$2^3\cdot 5^3$	$2^3\cdot 5^3$

$\xrightarrow{\text{sum}} (1+2^1+2^2+2^3+2^4)$

$\longrightarrow (1+2^1+2^2+2^3+2^4)\cdot 5$

$\longrightarrow (1+2^1+2^2+2^3+2^4)\cdot 5^2$

$\longrightarrow (1+2^1+2^2+2^3+2^4)\cdot 5^3$

$+$ (sum)

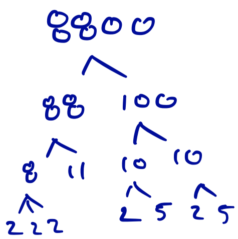
$(1+2^1+2^2+2^3+2^4)\cdot(1+5+5^2+5^3)$

$$\sigma(2000) = \sigma(2^5)\,\sigma(5^4) = 31\cdot 156 = \boxed{4836}$$

Theorem 2 If $n \in \mathbb{Z}^+$, and $n = p_1^{e_1}\cdots p_k^{e_k}$ is its prime-power decomposition, then

$$\sigma(n) = \sigma(p_1^{e_1}) \cdots \sigma(p_k^{e_k}).$$

Ex Compute σ of 8800



$$8800 = 2^5 \cdot 5^2 \cdot 11$$

$$\sigma(8800) = \underbrace{\sigma(2^5)}_{\frac{2^6-1}{2-1}} \cdot \underbrace{\sigma(5^2)}_{1+5+5^2} \cdot \underbrace{\sigma(11)}_{1+11}$$

$$= 63 \cdot 31 \cdot 12 = \boxed{23436}$$

Theorem 4 The function σ is multiplicative.

Pf. see book.