Math 108—Outline for Exam 2

Definitions

One thing that I hope you all take away from this course is a fluency in the language of logic and set theory. To that end, you are expected to be able to recite the definitions for the following terms.

- the **power set** of a set S (symbolically $\mathcal{P}(S)$)
- the **Cartesian product** of sets X and Y (symbolically $X \times Y$)
- a **relation** on a set X
- the **reflexive**, **symmetric**, and **transitive** properties for a relation $\sim$ on a set X
- the **set of relatives of** x when $\sim$ is a relation on A (symbolically $[x]$)
- an **equivalence relation** on a set X
- a **partition** of a set A

Problems to Practice Redo the exercises from the notes!

1. Working with Cartesian products
2. Working with unions and intersections over families of sets
3. Checking if a relation is reflexive, symmetric, and transitive
4. Working with equivalence relations—for example, determining $[x]$ (the equivalence class of an element)
5. Checking if a collection of subsets of A is a partition of A
6. "Example or no such example exists" problems
   (a) Examples of elements, sets, and/or relations meeting certain criteria
   (b) Practice "prove or disprove" problems

Proofs Redo the theorems from the notes!

1. Practice proving theorems with Induction
2. Practice proving (or disproving) the reflexive, symmetric, and transitive properties for a relation
3. Practice proving properties about equivalence relations
4. Re-practice the "basics," for example: how to prove implications, how to prove that a set is contained in another set, or how to prove that two sets are equal

Extra Practice Make sure to redo the exercises and theorems from the notes first!

1. Let $A = \{1, 2, 3\}$. Give an example (if one exists) of sets B, C, and D such that the following are true. If no example exists, simply write "not possible." *The notation $|X|$ means the number of elements in the set X.*

   (a) $D \subseteq A \times A$, $|D| = 4$
   (b) $D \subseteq A \times A$, $D \subseteq A$
   (c) $C \times C \subseteq A \times B$, $|A \times B| = 15$
   (d) $C \times C \subseteq A \times B$, $|C \times C| = 15$
   (e) $D \subseteq \mathcal{P}(A)$, $|D| = 4$
   (f) $D \in \mathcal{P}(A)$, $|D| = 4$
   (g) $D \subseteq \mathcal{P}(A \times A)$, $|D| = 4$
   (h) $D \in \mathcal{P}(A \times A)$, $|D| = 4$

2. For each $n \in \mathbb{N}$, let A_n be the interval of real numbers $A_n = (-n, 2 + \frac{1}{n})$.

   (a) Find $A_1 \cap A_2$.
   (b) Find $\bigcap_{n \in \mathbb{N}} A$.
   (c) Find $A_1 \cup A_2$.
   (d) Find $\bigcup_{n \in \mathbb{N}} A$.

3. Give an example of each of the following:

   (a) infinitely many *open* intervals $A_1, A_2, \ldots$ of real numbers such that $\bigcap_{n=1}^{\infty} A = [-2, 2]$.

   (b) infinitely many *pairwise disjoint* intervals $B_1, B_2, \ldots$ of real numbers such that $\bigcup_{n=1}^{\infty} B_n = \mathbb{R}^+$.

   ($\mathbb{R}^+$ denotes the positive real numbers.)

4. Determine if each of the following relations are reflexive, symmetric, or transitive.

- If you believe the relation has a property, you can just say so, without proof.
- If you believe the relation does not have a property, give an example showing that the property fails.

(a) The relation R on $\mathbb{R}$ defined by $x \, R \, y \iff | x - y | < 1$.
(b) The relation R on $\mathbb{R}$ defined by $x \, R \, y \iff \exists z \in \mathbb{R}$ such that $x - y = z^2$.
(c) The relation R on $\mathbb{Z}$ defined by $x \, R \, y \iff x^2 + y^2$ is even.

5. Determine if P is a partition of A.

- If you believe that P is a partition, you can just say so, without proof.
- If you believe that P is not a partition, explain why not.

(a) A is the set of positive real numbers, $P = \{A_k \mid k \in \mathbb{N}\}$ where A_k is the interval $A_k = \left(\frac{1}{k}, k\right]$.
(b) $A = \mathbb{R} \times \mathbb{R}$, $P = \{A_k \mid k \in \mathbb{R}\}$ where $A_k = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y + x = k\}$.

6. Consider the relation R on $\mathbb{Z}^+$ defined by $x \sim y \iff$ "x divides y."

(a) Find the set of all y such that $2 \sim y$.
(b) Find the set of all y such that $5 \sim y$.
(c) Find the set of all x such that $x \sim 30$.
(d) Find the set of all x such that $x \sim 17$.

7. Notice that every $x \in \mathbb{Z}^+$ can be written as $x = 2^k n$ where n is an odd number. This is saying that 2 appears k-times in the prime factorization of x. Define $\nu(x) = k$. (Just to check, $\nu(4) = 2$ and $\nu(60) = \nu(2^2 \cdot 3 \cdot 5) = 2$.) Now define a relation $\sim$ on $\mathbb{Z}^+$ by $x \sim y \iff \nu(x) = \nu(y)$, and notice that $\sim$ is an equivalence relation.

(a) List five different elements in [4] (the equivalence class of 4). Note that we already saw $4 \sim 60$.
(b) Find an element less than 10 in [168] (the equivalence class of 168).

8. Prove that for all sets A, B, C and D, if $C \subseteq A$ and $D \subseteq B$, then $D - A \subseteq B - C$.

9. Use induction to prove $\dfrac{1}{2!} + \dfrac{2}{3!} + \cdots + \dfrac{n}{(n+1)!} = 1 - \dfrac{1}{(n+1)!}$ for all $n \in \mathbb{N}$.

10. Use induction to prove that $3^n \geq 1 + 2^n$ for all $n \in \mathbb{N}$.

11. Let f_n be the n^{th} Fibonacci number. Recall that the Fibonacci numbers are defined by

$$f_1 = 1,\ f_2 = 1, \text{ and } f_n = f_{n-1} + f_{n-2} \text{ for } n \geq 2.$$

Prove that $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$ for all $n \geq 1$.

12. Let A and B be sets, and let f be any function from A to B. The *kernel* of f is defined to be the relation on A given by

$$x \, R \, y \iff f(x) = f(y).$$

Prove that the kernel of f is an equivalence relation on A.