CHAPTER 3

COMPLETNESS
&
COMPACTNESS

3.2 Completeness

Recall:

(soundness) If $\Sigma \vdash \phi$, then $\Sigma \models \phi$

We now aim for the converse.

of our deductive system

Thm (completeness) If $\Sigma \models \phi$, then $\Sigma \vdash \phi$.

the def. of a complete ded. system

Gödel – 1929 Dissertation

THE PROOF BEGINS

- Fix Σ and ϕ.
- We will assume that $\mathcal{L}$ is countable — this implies that the $\mathcal{L}$-formulas can be enumerated in an infinite list: $\alpha_1, \alpha_2, \ldots$

Step 0 Rephrasing the problem.

(a) we may assume ϕ is a sentence.

why? ... Prop 2.7.2 says $\Sigma \vdash \phi \leftrightarrow \Sigma \vdash \forall x \phi$.
Repeating this for all free variables in ϕ, say $x_1, \ldots, x_n$, $\Sigma \vdash \phi \leftrightarrow \Sigma \vdash \underbrace{\forall x_1, \ldots, x_n \phi}_{\text{sentence}}$

(b) we may assume Σ consists of sentences
why? ... Prop 2.7.3 (sim. to before).

Def: $\bot :\equiv \forall x (x = x) \wedge \neg \forall x (x = x)$. Also, we say Σ is consistent if $\Sigma \nvdash \bot$.

(C) We may assume ϕ is $\perp$

Why? ... Suppose we can prove $\Sigma \models \perp \to \Sigma \vdash \perp$.

Let's prove $\Sigma \models \phi \to \Sigma \vdash \phi$ from this.

$$\Sigma \models \phi \implies \Sigma \cup (\neg \phi) \models \perp$$ (b/c there are no models of $\Sigma \cup (\neg\phi)$)

$$\implies \Sigma \cup (\neg \phi) \vdash \perp$$

$$\implies \Sigma \vdash \phi$$ "proof by cont." Exerc. 2.7.1 #4

<u>Recap:</u> We will prove the completeness Thm if we can prove:

$$\Sigma \models \perp \to \Sigma \vdash \perp$$

for Σ any set of sentences. Looking at the contrapositive we find that

To prove the completeness Thm, it suffices to prove that

if Σ is a consistent set of sentences, then Σ has a model.

Let Σ be a consistent set of $\mathcal{L}$-sentences. We must construct a model.

$\mathcal{L} \cup \{\text{lots of constants}\}$

Rough Outline

$\boxed{\text{Step 1}}$ We enlarge both $\mathcal{L}$ and Σ to $\mathcal{L}'$ and Σ'.

$\boxed{\text{Step 2}}$ We build a model $\mathfrak{M}$ of Σ' in $\mathcal{L}'$. The universe is essentially the set of variable-free terms of $\mathcal{L}'$.

$\boxed{\text{Step 3}}$ We restrict $\mathfrak{M}$ to $\mathcal{L}$.

THE PROOF CONTINUES

$\boxed{\text{Step 1}}$ Let $\mathcal{L}_0 = \mathcal{L}$. Define

Henkin constants

1.1 $$\boxed{\mathcal{L}_1 = \mathcal{L}_0 \cup \{c_1, c_2, c_3, \dots\}}$$

where each c_i is a constant symbol not already in $\mathcal{L}_0$.

Lemma 3.2.3 Σ is still consistent as $\mathcal{L}_1$-sentences.

pf idea

Suppose not. Let $D_1 = (\alpha_1, \dots, \alpha_m)$ be a ded. of $\vdash$ with α_i an $\mathcal{L}_1$-formula.

$\perp$

- $\exists k$ s.t. the new constants occurring in D_1 are among $c_1, \dots, c_k$
- Let $v_1, \dots, v_k$ be variables not occuring in D_1.
- Define $D_0 = \left((\alpha_1)^{c_1, \dots, c_k}_{v_1, \dots, v_k}, \dots, (\alpha_m)^{c_1, \dots, c_k}_{v_1, \dots, v_k} \right)$

still $\perp$

- A little work shows $D_0 \vdash \perp$, but now D_0 is in $\mathcal{L}_0$. Contradiction.

We now expand Σ. First, list all $\mathcal{L}_0$-sentences of the form $\exists x \theta$:

$$\exists x_1 \theta_1, \ \exists x_2 \theta_2, \dots$$

e.g. $\underbrace{\exists v_7}_{x_1} \underbrace{(v_7 < v_7)}_{\theta_1}$, $\underbrace{\exists v_3}_{x_2} \underbrace{(\exists v_5 (v_3 = v_5))}_{\theta_2}$, ...

Define

- $\psi_i := (\exists x_i \theta_i) \to (\theta_i)^{x_i}_{c_i}$ ← Henkin axioms "c_i is a witness for $\exists x_i \theta_i$"
- $H_1 = \{\psi_i \mid i \geq 1\}$

1.2 Now, let $\Sigma_0 = \Sigma$, and define $\boxed{\Sigma_1 = \Sigma_0 \cup H_1}$.

<u>Lemma 3.2.4</u> Σ_1 is still consistent (as $\mathcal{L}_1$-sent.)

<u>pf idea</u>

Suppose not. Let m be smallest integer s.t. $\Sigma \cup \{\psi_1, \dots, \psi_m, \psi_{m+1}\}$ is inconsistent (which exists since deductions are finite). Now

$$\Sigma \cup \{\psi_1, \dots, \psi_{m+1}\} \vdash \bot$$

$\neg \psi_m \vee \bot$

$$\Longrightarrow \Sigma \cup \underbrace{\{\psi_1, \dots, \psi_m\}}_{A} \vdash (\psi_{m+1} \to \bot)$$

Ded. Theorem

$$\Longrightarrow \Sigma \cup A \vdash \neg \psi_{m+1}$$

(ψ_{m+1} is of form $\exists x \theta \to \theta^x_c$)

$$\Longrightarrow \Sigma \cup A \vdash \exists x \theta \wedge \neg \theta^x_c$$

$$\Longrightarrow \boxed{\Sigma \cup A \vdash \neg \forall x \neg \theta} \ \star$$

and

$$\Sigma \cup A \vdash \neg \theta^x_c$$

$$\Longrightarrow \Sigma \cup A \vdash \neg \theta^x_z$$

for z a new variable (as in prev. proof)

$\Longrightarrow \quad \Sigma \cup A \vdash \forall z \neg \theta^x_z$ Lem 2.7.2

($\forall z \neg \theta^x_z \to \neg(\theta^x_z)^z_x$ is a Q1 axiom)

$\Longrightarrow \quad \Sigma \cup A \vdash \neg(\theta^x_z)^z_x = \neg \theta$

$\Longrightarrow \quad \boxed{\Sigma \cup A \vdash \forall x \neg \theta}$ ☆↯ Lem 2.7.2

$\Longrightarrow \quad \Sigma \cup A \vdash \perp$ by ☆ and ☆☆

$\Longrightarrow \quad \Sigma \cup \{\varphi_1, \dots, \varphi_m\}$ is inconsistent,

but $m+1$ was least such.

Thus Σ_1 is consistent. □

1.3 Repeat ← extend with ∞-many new constants each time

- $\mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \cdots$
- $\Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \cdots$ ← extend with Henkin axioms H_2 for the $\mathcal{L}_1$-sentences

Each Σ_i remains consistent, with same proof as before.

- $\mathcal{L}' = \bigcup_{i \in \mathbb{N}} \mathcal{L}_i$
- $\hat{\Sigma} = \bigcup_{i \in \mathbb{N}} \Sigma_i$

(1.4) Further extend $\hat{\Sigma}$ to Σ' to ensure either $\sigma \in \Sigma'$ or $\neg\sigma \in \Sigma'$ for all $\mathcal{L}'$-sentences σ.

- Enumerate the $\mathcal{L}'$-sentences:

$$\sigma_1, \sigma_2, \ldots$$

- $\Sigma^0 = \hat{\Sigma}$
- $\Sigma^{k+1} = \begin{cases} \Sigma^k \cup \{\sigma_k\} & \text{if } \Sigma^k \cup \{\sigma_k\} \text{ is cons.} \\ \Sigma^k \cup \{\neg\sigma_k\} & \text{o/w.} \end{cases}$
- $\Sigma' = \bigcup_{k \in \mathbb{N}} \Sigma^k$

<u>Lemma</u> For every formula σ, $\sigma \in \Sigma'$ or $\neg\sigma \in \Sigma'$. Also, Σ' is consistent (as are each Σ^k).

<u>pf</u> exercise.

<u>Lemma 3.2.5</u> If σ is a Σ' sentence, then

$$\sigma \in \Sigma' \text{ iff } \Sigma' \vdash \sigma$$

<u>pf</u> ($\Longrightarrow$) clear. So, assume $\Sigma' \vdash \sigma$. If $\sigma \notin \Sigma'$, the $\neg\sigma \in \Sigma'$. Thus, $\Sigma' \vdash \sigma$ and $\Sigma' \vdash \neg\sigma$ so $\Sigma' \vdash \bot$. But Σ' is consistent. $\square$

Step 2 Constructing a model of Σ' (in $\mathcal{L}'$)

<u>Idea</u>: the constant symbols are kind of the elements of the universe. But, what when we use functions: we need $c_1 + c_7$ to again be in our universe. Okay, so instead of constants, let's use variable-free terms. Close. what if

$$c_1 + c_7 = c_8$$

is in Σ'? we need $c_1 + c_7$ to be the same element as c_8, but it can't be b/c they are different seq. of symbols. So...

Let T be the set of variable free $\mathcal{L}'$-terms.

Define $\boxed{t_1 \sim t_2 \longleftrightarrow (t_1 = t_2) \in \Sigma'.}$

<u>Lemma</u> $\sim$ is an equivalence rel. on T.

<u>Pf</u> book + exercises.

<u>universe of $\mathcal{M}$</u>

$M = T/\sim$, i.e. M is the set of $\sim$-equivalence classes of T. We write these as $[t]$ for $t \in T$.

<u>Ex</u> if $+$ is a binary funct. sym. in $\mathcal{L}'$ and c_1, c_7 are constants in $\mathcal{L}'$, then

$$[c_1], [c_7], [c_1 + c_7] \in M.$$

Also, if $c_1 + c_7 = c_8 \in \Sigma'$, then $[c_1 + c_7] = [c_8]$.

symbols in language

actual elements in M.

<u>Constants</u> $c^{\mathcal{M}} = [c]$.

<u>Functions</u> $f^{\mathcal{M}}([t_1], \ldots, [t_n]) = [f(t_1, \ldots, t_n)]$

<u>Ex</u> $[c_1] +^{\mathcal{M}} [c_7] = [c_1 + c_7]$

! Is this well-defined? It is — let's just look at an example.

Ex Suppose $\mathcal{L}_{NT} \subseteq \mathcal{L}'$.

Let $t_1 = c_1 + c_7$, $t_2 = c_8$ and $t_1 = t_2 \in \Sigma'$.

Let's look at the function S. We have

$$[t_1] = [t_2] \qquad (\text{since } t_1 = t_2 \in \Sigma')$$

and we want to show

$$S^{\mathfrak{M}}([t_1]) = S^{\mathfrak{M}}([t_2]) \quad (\text{i.e. that } St_1 = St_2 \in \Sigma').$$

By Lem. 3.2.5, it suffices to show $\Sigma' \vdash St_1 = St_2$.

$$x = y \to Sx = Sy \qquad E2$$

$$\vdots$$

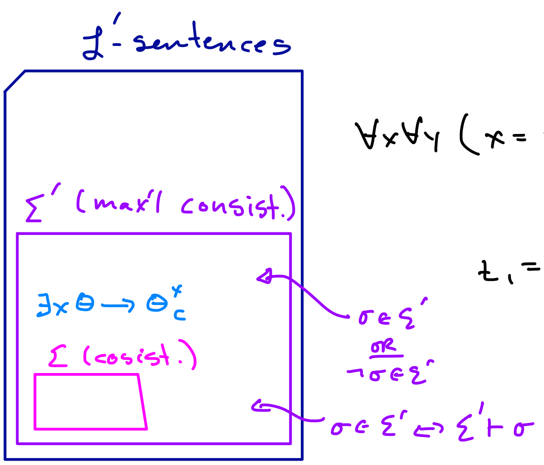
$$\forall x \forall y (x = y \to Sx = Sy) \qquad 2.7.2$$

$$\vdots$$

$$t_1 = t_2 \to St_1 = St_2 \qquad Q1 \text{ (twice)}$$

$$t_1 = t_2 \qquad \Sigma'$$

$$St_1 = St_2 \qquad PC$$



Relations $R^{\mathfrak{M}}([t_1], \ldots, [t_n])$ iff $Rt_1 \ldots t_n \in \Sigma'$.

! Again, we need to show this is well defined.

Recall

Prop 3.2.6 $\mathfrak{M} \models \Sigma'$!!

we prove $\sigma \in \Sigma'$ iff $\mathfrak{M} \models \sigma$. (for all sentences σ)

pf

We proceed by induction on the complexity of σ.

* Recall def. of $\mathfrak{M}$.

A useful fact: if s is a vaf into $\mathcal{M}$ and t is a var.-free term, then $\bar{s}(t) = [t]$.

∘ need to think about vaf's … not too hard but uses def. of $f^{\mathcal{M}}$ and $c^{\mathcal{M}}$

① $\sigma :\equiv t_1 = t_2$ where t_1, t_2 are variable-free terms since σ is a sentence

Then,

$$\sigma \equiv t_1 = t_2 \in \Sigma' \text{ iff } t_1 \sim t_2$$
$$\text{iff } [t_1] = [t_2]$$
$$\text{iff } \bar{s}(t_1) = \bar{s}(t_2) \text{ for all vaf into } \mathcal{M}$$
$$\text{iff } \mathcal{M} \models t_1 = t_2$$

② $\sigma :\equiv R(t_1, \ldots, t_n)$ where $t_1, \ldots, t_n$ are var. free

Then

$$\sigma \equiv R(t_1, \ldots, t_n) \in \Sigma' \text{ iff } ([t_1], \ldots, [t_n]) \in R^{\mathcal{M}} \quad \text{by def. of } R^{\mathcal{M}}$$
$$\text{iff } (\bar{s}(t_1), \ldots, \bar{s}(t_n)) \in R^{\mathcal{M}}$$
$$\text{iff } \mathcal{M} \models R(t_1, \ldots, t_n).$$

③ $\sigma :\equiv \neg\alpha$ where $\alpha \in \Sigma'$ iff $\mathcal{M} \models \alpha$ (by induction)

Then,

$$\sigma \in \Sigma' \text{ iff } \alpha \notin \Sigma'$$
$$\text{iff } \mathcal{M} \not\models \alpha$$
$$\text{iff } \mathcal{M} \models \neg\alpha$$
$$\text{iff } \mathcal{M} \models \sigma.$$

④ $\sigma :\equiv \alpha \vee \beta$

similar.

(5) $\sigma :\equiv \forall x \phi$.

($\Rightarrow$) Assume $\sigma \in \Sigma'$. WTS $\mathcal{M} \models \sigma$.

$$
\begin{aligned}
\mathcal{M} \models \sigma \quad & \text{iff} \quad \mathcal{M} \models \forall x \phi \\
& \text{iff} \quad \mathcal{M} \models \forall x \phi[s] \quad \text{for any vaf } s \\
& \text{iff} \quad \mathcal{M} \models \phi\big[s[x|m]\big] \quad \text{for any } m \in \mathcal{M}. \\
& \text{iff} \quad \mathcal{M} \models \phi\big[s[x|[t]]\big] \quad \text{for any var. free term } t \\
& \text{iff} \quad \mathcal{M} \models \phi\big[s[x|\bar{s}(t)]\big] \quad \text{by the useful fact} \\
& \text{iff} \quad \mathcal{M} \models \phi^x_t[s] \quad \text{by Thm 2.6.2 (a tech. result for soundness)} \\
& \text{iff} \quad \mathcal{M} \models \phi^x_t \quad \text{since } \phi^x_t \text{ is var. free.} \\
& \text{iff} \quad \phi^x_t \in \Sigma' \quad \text{by induction} \\
& \text{iff} \quad \Sigma' \vdash \phi^x_t \quad \text{Lemma 3.2.5}
\end{aligned}
$$

fewer quantifiers than σ!!

Now, $\sigma \in \Sigma'$ so $\Sigma' \vdash \forall x \phi$. By (Q1), $\Sigma' \vdash \forall x \phi \to \phi^x_t$. (Note t is var. free so certainly sub. for x.) Thus $\Sigma' \vdash \phi^x_t$, so $\mathcal{M} \models \sigma$.

($\Leftarrow$) Assume $\sigma \notin \Sigma'$. WTS $\mathcal{M} \not\models \sigma$.

$$
\begin{aligned}
\sigma \notin \Sigma' & \Rightarrow \neg\sigma \in \Sigma' \\
& \Rightarrow \Sigma' \vdash \neg\sigma \quad \text{Lem. 3.2.5} \\
& \Rightarrow \Sigma' \vdash \neg\forall x \phi \\
& \Rightarrow \Sigma' \vdash \exists x \neg\phi \\
& \Rightarrow \exists x \neg\phi \in \Sigma' \\
& \Rightarrow \exists x \neg\phi \to \neg\phi^x_c \in \Sigma' \quad \text{for some Henkin constant } c
\end{aligned}
$$

Note $\neg\sigma \equiv \neg\forall x \phi$

$\neg\forall x \neg\neg\phi$

$$\Rightarrow \Sigma' \vdash \exists x \neg\phi \wedge (\exists x \neg \phi \rightarrow \neg \phi^x_c)$$ Lemma 3.2.5

$$\Rightarrow \Sigma' \vdash \neg \phi^x_c$$

$$\Rightarrow \neg \phi^x_c \in \Sigma'$$

$$\Rightarrow \phi^x_c \notin \Sigma'$$

$$\Rightarrow \mathcal{M} \nvDash \phi^x_c$$ by induction — ϕ^x_c has less quantifiers

$$\Rightarrow \mathcal{M} \nvDash \forall x \phi$$

$\square$

$\boxed{\text{Step 3}}$ Restrict $\mathcal{M}$ to $\mathcal{L}$.

we know that $\mathcal{M} \vDash \Sigma'$ and $\Sigma \subseteq \Sigma'$. Also $\mathcal{M}$ is an $\mathcal{L}'$-structure. Write $\mathcal{M}|_{\mathcal{L}}$ for $\mathcal{M}$ viewed as an $\mathcal{L}$-structure (just forget about the extra constant symbols... but the elements are still in $\mathcal{M}|_{\mathcal{L}}$). Then, it's not hard to see that $\mathcal{M}|_{\mathcal{L}} \vDash \Sigma$.

THE PROOF ENDS

3.3 Compactness (A.K.A. Dessert)

Theorem (Compactness) Let Σ be any set of formulas. Then, Σ has a model iff every finite subset of Σ has a model.

Pf

($\Longrightarrow$) If $\mathcal{M} \models \Sigma$, then certainly $\mathcal{M} \models \Sigma_0$ for every $\Sigma_0 \subseteq \Sigma$.

($\Longleftarrow$) Suppose every finite subset of Σ has a model (which may be different for different subsets.

We argue by contradiction — assume Σ has no model. Then $\Sigma \models \bot$, and

$$\Sigma \models \bot \Longrightarrow \Sigma \vdash \bot \qquad \text{(completeness)}$$

$$\Longrightarrow \Sigma_0 \vdash \bot \qquad \text{for some } \underline{\text{finite}}\ \Sigma_0 \subseteq \Sigma \text{ b/c } \boxed{\text{deductions are finite}}$$

$$\Longrightarrow \Sigma_0 \models \bot \qquad \text{(soundness)}$$

$$\Longrightarrow \Sigma_0 \text{ has no model}$$

$$\Longrightarrow\Longleftarrow$$

$\square$

later

Cor. 3.3.2 $\Sigma \models \theta$ iff there is a finite $\Sigma_0 \subseteq \Sigma$ s.t. $\Sigma_0 \models \theta$.

Pf

$$\Sigma \models \theta \Longleftrightarrow \Sigma \vdash \theta$$

$$\Longleftrightarrow \Sigma_0 \vdash \theta \text{ for some finite } \Sigma_0 \subseteq \Sigma$$

deductions are finite

clear

$$\Longleftrightarrow \Sigma_0 \models \theta \text{ for some finite } \Sigma_0 \subseteq \Sigma$$

$\square$

$\boxed{\text{Application 1}}$ the property of "being finite" is not a first order property!

<u>Ex</u> Let $\mathcal{L}_G = \{\cdot, {}^{-1}, 1\}$ The axioms for a group are

binary $g \cdot h$ unary g^{-1} constant

$$\gamma_1 :\equiv (x \cdot y) \cdot z = x \cdot (y \cdot z)$$

$$\gamma_2 :\equiv x \cdot 1 = x \wedge 1 \cdot x = x$$

$$\gamma_3 :\equiv x \cdot x^{-1} = 1 \wedge x^{-1} \cdot x = 1$$

$\boxed{\text{Question 1}}$: is there a set of formulas Σ such that

$G \models \Sigma$ iff G is a group?

<u>Answer 1</u>: yes of course, $\Sigma = \{\gamma_1, \gamma_2, \gamma_3\}$.

$\boxed{\text{Question 2}}$: is there a set of formulas Σ s.t.

$G \models \Sigma$ iff G is a group with at most 4 elements?

<u>Answer 2</u>: yes: $\Sigma = \{\gamma_1, \gamma_2, \gamma_3, \sigma\}$ wher

$$\sigma :\equiv \exists x_1, x_2, x_3, x_4 \, \forall y \, (y = x_1 \vee y = x_2 \vee y = x_3 \vee y = x_4)$$

$\boxed{\text{Question 3:}}$ is there a set of formulas Σ s.t.

$G \models \Sigma$ iff G is an infinite group?

<u>Answer 3</u> yes: $\Sigma = \{\gamma_1, \gamma_2, \gamma_3\} \cup \left\{\exists x_1 \dots x_k \left(\bigwedge_{1 \le i < j \le k} (x_i \neq x_j) \right) \middle| \, k \ge 2 \right\}$

<u>Question 3</u>: is there a set of formulas Σ s.t.

$G \models \Sigma$ iff G is a finite group?

Think!! ...

Suppose such a Σ does exist. Define

$$\alpha_2 :\equiv \exists x_1 \exists x_2 \,(x_1 \neq x_2)$$
$$\alpha_3 :\equiv \exists x_1 \exists x_2 \exists x_3 \left[(x_1 \neq x_2) \wedge (x_2 \neq x_3) \wedge (x_1 \neq x_3)\right]$$
$$\vdots$$

Note: $G \models \alpha_k$ iff G has at least k elements.

Define $\hat{\Sigma} = \Sigma \cup \{\alpha_k \mid k \geq 2\}$. We apply compactness...

Let $A \subseteq \hat{\Sigma}$ be finite. Let m be the largest integer s.t. $\alpha_m \in A$. Let C_m be the cyclic group with m elements. Then

- $C_m \models \alpha_k$ for all $k \leq m$
- $C_m \models \Sigma$ (by assumption)

Since, $A \subseteq \Sigma \cup \{\alpha_1, \ldots, \alpha_k\}$ and C_m models the RHS, we find $C_m \models A$. Thus, every finite subset of $\hat{\Sigma}$ has a model, so by compactness, $\hat{\Sigma}$ has some model $\hat{G}$. As $\hat{G} \models \alpha_k$ for all $k \geq 2$, $\hat{G}$ is infinite. But also, $\Sigma \subseteq \hat{\Sigma}$, so $\hat{G} \models \Sigma$, a contradiction.

Thus, the finite groups can not be axiomatized.

Ex Let $\mathcal{L}_0 = \{<\}$. The axioms for a linear order are

$$L1 :\equiv \forall x \forall y \, (x<y \vee x=y \vee y<x)$$
$$L2 :\equiv \forall x \, \neg(x<x)$$
$$L3 :\equiv \forall x \forall y \forall z \, [(x<y \wedge y<z) \longrightarrow x<z]$$

Then, there is no set of axioms Σ s.t.

$$\mathcal{M} \models \Sigma \text{ iff } \mathcal{M} \text{ is a finite linear order.}$$

pf you do this... follow previous example.

Thm Suppose Σ is a set of formulas s.t. Σ has models of arbitrarily large finite order. Then Σ has an infite model.

pf you do this... follow previous example.

In other words, you can not axiomitize the property of being finite.

Application 2

You can not axiomitize a single structure (in a 1^{st} order way).

Ex Let's think about $\mathbb{N}$ interpreted in the usual way w.r.t. $\mathcal{L}_{NT}$.

<u>Question 1</u>: Is there a set of formulas Σ

s.t.
$$\mathfrak{M} \models \Sigma \text{ iff } \mathfrak{M} \cong \mathbb{N}?$$

Think... what is the most restrictive Σ we could try?

<u>Def</u> Let $\mathfrak{M}$ be an $\mathcal{L}$-structure. The <u>theory of $\mathfrak{M}$</u> is $Th(\mathfrak{M}) = \{\phi \mid \mathfrak{M} \models \phi$ for ϕ an $\mathcal{L}$-form.$\}$.

what if we use $\Sigma = Th(\mathbb{N})$? ... we should have a chance ... right?

Suppose Σ exists; so, $\mathfrak{M} \models \Sigma$ iff $\mathfrak{M} \cong \mathbb{N}$.

Expand $\mathcal{L}_{NT}$ to $\mathcal{L} = \mathcal{L}_{NT} \cup \{c\}$. Let Γ be the following set of formulas:

$$\alpha_0 :\equiv \quad 0 < c$$
$$\alpha_1 :\equiv \quad \bar{1} = S0 < c$$
$$\alpha_2 :\equiv \quad \bar{2} \; SS0 < c$$
$$\vdots$$

<u>Claim</u>: Every finite subset of $\Sigma \cup \Gamma$ has a model.

<u>Pf</u> Let $A \subseteq \Sigma \cup \Gamma$ be finite.

- let k be largest s.t. $\alpha_k \in A$.
  - thus $A \subseteq \Sigma \cup \{\alpha_0, \dots, \alpha_k\}$
- Make $\mathbb{N}$ an $\mathcal{L}$-structure $\tilde{\mathbb{N}}$ by defining $c^{\tilde{\mathbb{N}}} = k+1$
  - thus, $0^{\tilde{\mathbb{N}}} <^{\tilde{\mathbb{N}}} c^{\tilde{\mathbb{N}}}, \bar{1}^{\tilde{\mathbb{N}}} <^{\tilde{\mathbb{N}}} c^{\tilde{\mathbb{N}}}, \dots, \bar{k}^{\tilde{\mathbb{N}}} <^{\tilde{\mathbb{N}}} c^{\tilde{\mathbb{N}}}$
  - so, $\tilde{\mathbb{N}} \models \{\alpha_0, \dots, \alpha_k\}$
  - Also, $\tilde{\mathbb{N}} \models \Sigma$ (by assumption)

- Thus $\mathbb{N} \models A$.

$\square$

By compactness, $\Sigma \cup \Gamma$ has a model, say $\mathcal{M}$. Notice that $\mathcal{M} \models \Sigma$, and $\bar{n} < c^{\mathcal{M}}$ for all $n \in \mathbb{N}$. No such element like this exists in $\mathbb{N}$, so $\mathcal{M} \not\cong \mathbb{N}$. Thus, No there is no set Σ of formulas, s.t.

$$\mathcal{M} \models \Sigma \text{ iff } \mathcal{M} \cong \mathbb{N}.$$

$\square$

<u>Def</u> If $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures, we say that $\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent if $Th(\mathcal{M}) = Th(\mathcal{N})$. We denote this by

$$\mathcal{M} \equiv \mathcal{N}$$

* we just saw that $\mathcal{M} \equiv \mathbb{N} \not\Rightarrow \mathcal{M} \cong \mathbb{N}$.

In fact, a different construction (using compactness) can be used to show that...

<u>Theorem</u> If $\mathcal{N}$ is infinite, then there exists structures $\mathcal{M}$ (of arbitrarily large cardinality) s.t. $\mathcal{M} \equiv \mathcal{N}$ but $\mathcal{M} \not\cong \mathcal{N}$.

Application 3

Creating superstructures with special elements.

Ex Constructing hyperreals.

Let $\mathcal{L}_{OR} = \{+, \cdot, 0, 1, <\}$ (ordered ring)

The Goal: Create an $\mathcal{L}_{OR}$-structue $\mathbb{R}^*$ s.t.

① $\mathbb{R} \subseteq \mathbb{R}^*$

② $\mathbb{R} \equiv \mathbb{R}^*$

③ $\mathbb{R}^*$ contains infinitesimal elements, i.e. there is an $\varepsilon \in \mathbb{R}^*$ s.t.

$0 < \varepsilon < r$ for every $r \in \mathbb{R}$.

- Expand $\mathcal{L}_{OR}$ to $\mathcal{L}_{\mathbb{R}} = \mathcal{L}_{OR} \cup \{c_r \mid r \in \mathbb{R}\}$.
  - make $\mathbb{R}$ an $\mathcal{L}_{\mathbb{R}}$-st. by defining

$$c_r^{\mathbb{R}} = r.$$

  - let $\Sigma_1 = Th(\mathbb{R})$ in $\mathcal{L}_{\mathbb{R}}$.
    - * thus $\forall x (x \neq 0 \rightarrow \exists y (xy = 1)) \in \Sigma_1$
    - * and also $c_3 < c_\pi \in \Sigma_1$
    - * can you give me another?

- Add one more constant, which will ultimately point to an infitesimal element.

$$\mathcal{L} = \mathcal{L}_{\mathbb{R}} \cup \{a\}$$

- Let Γ be the following (very large) set of sentences

$$\Sigma_2 = \{ \underbrace{0 < a \wedge a < c_r}_{\alpha_r} \mid r \in \mathbb{R}, r > 0 \}$$

<u>Claim</u>: Every finite subset of $\Sigma_1 \cup \Sigma_2$ has a model.

<u>pf</u> Let $A \subseteq \Sigma_1 \cup \Sigma_2$ be finite.

- there is a smallest $r_0 \in \mathbb{R}$ s.t. $\alpha_{r_0} \in A$
  - thus $A \subseteq \Sigma_1 \cup \{0 < a < r \mid r \geq r_0\}$
- make $\mathbb{R}$ an $\mathcal{L}$-structure $\tilde{\mathbb{R}}$ by defining

$$a^{\tilde{\mathbb{R}}} = \frac{r_0}{2}$$

  - thus $0^{\tilde{\mathbb{R}}} < a^{\tilde{\mathbb{R}}} < r_0 \leq r^{\tilde{\mathbb{R}}}$ for all $r \geq r_0$
  - thus $\tilde{\mathbb{R}} \models \alpha_r$ for all $r \geq r_0$
  - also $\tilde{\mathbb{R}} \models \Sigma_1 = Th(\mathbb{R})$
- Thus $\tilde{\mathbb{R}} \models A$. $\square$

By compactness, $\Sigma_1 \cup \Sigma_2$ has a model, which we call $\mathbb{R}^*$. Note that

① $\mathbb{R} \subseteq \mathbb{R}^*$ (kind of... if we identify $r \longleftrightarrow c_r^{\mathbb{R}^*}$

② $\mathbb{R}^* \equiv \mathbb{R}$ since $\mathbb{R}^* \models Th(\mathbb{R})$

③ $\mathbb{R}^*$ contains the infintesimal element $a^{\mathbb{R}^*}$, since $\mathbb{R}^* \models \{0 < a < c_r \mid r \in \mathbb{R}, r > 0\}$

which we think of as just r.

Done!

<u>Def</u> Any element $a \in \mathbb{R}^*$ satisfying $0 < a < r \;\forall r \in \mathbb{R}$ is called an <u>infinitesimal</u> element.

__Ex__ Let $s, t \in \mathbb{R}$. Then $r = s$ iff $|s-t| = a$ for some infinitesimal $a \in \mathbb{R}^*$.

__Pf__

$|s-t|$ infinitesimal iff $|s-t| < r$ for all $r \in \mathbb{R}$

iff $|s-t| = 0$ (since $s-t \geq 0$).

iff $s = t$ $\square$

* So two real #s are equal iff they are infinitesimally close.

__Def__ Any element $\alpha \in \mathbb{R}^*$ with $\alpha > r$ for all $r \in \mathbb{R}$ is called __infinite__.

__Ex__ $a \in \mathbb{R}$ is infinitesimal iff a^{-1} is infinite.

! ... and explain why a^{-1} exists.

① $\mathbb{R}$ models $\forall x\,(x \neq 0 \to \exists y\,(xy = 1))$.

Since $\mathbb{R}^* \equiv \mathbb{R}$, $\mathbb{R}^*$ models this too!

② $\mathbb{R}$ models $\forall x \forall y\,(0 < x < y \leftrightarrow 0 < y^{-1} < x^{-1})$

Since $\mathbb{R}^* \equiv \mathbb{R}$, $\mathbb{R}^*$ models this too!

Thus

$0 < a < r$ for all $r \in \mathbb{R}$, $r > 0$ iff $0 < r^{-1} < a^{-1}$ for all $r \in \mathbb{R}$, $r > 0$

iff $0 < s < a^{-1}$ for all $s \in \mathbb{R}$, $s > 0$.

$\square$