CHAPTER 4

Incompleteness

what & How

Let ϕ be any $\mathcal{L}_{NT}$-formula.

The Question (take 1)

If ϕ is a true statement about $\mathbb{N}$, can we prove it?

* Prove it? Starting from what? What are our starting axioms?

- Notice that if we use the axioms N, then we already saw that

$$\mathbb{N} \models \neg(x < x) \quad \text{but} \quad N \nvdash \neg(x<x)$$

Thus, N is <u>not strong enough</u> to prove all true statements about $\mathbb{N}$.

- However, if we use the axioms $Th(\mathbb{N})$, then clearly

$$\mathbb{N} \models \phi \quad \text{iff} \quad Th(\mathbb{N}) \vdash \phi.$$

So, $Th(\mathbb{N})$ is strong enough, but if you think about it, it's <u>too strong</u>. We don't know what's in $Th(\mathbb{N})$, e.g. is $\varphi :\equiv$ "every $x > 1$ is the sum of two primes" in $Th(\mathbb{N})$?

- Also if the axioms are inconsistent, then they can prove anything.

"Def" A set A of formulas is recursive or decidable if there exists an algorithm that can decide whether or not an arbitrary ϕ is in A or not.

* This is a notion of "not too complicated".

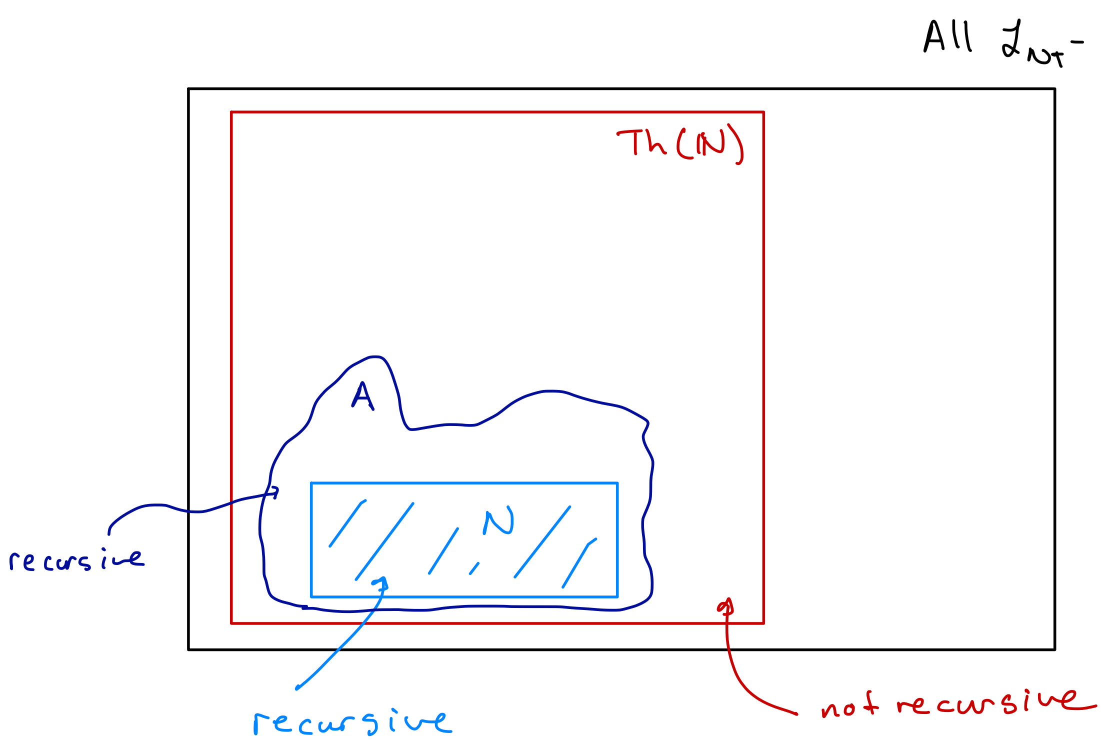
Ex The axiom set N is recursive but Th(N) is not.

The Question (take 2)

not too complicated ... but can be more complicated than N.

Does there exist a consistent and $\boxed{\text{recursive set}}$ of formulas A such that

$$\mathbb{N} \models \phi \text{ implies } A \vdash \phi \text{ ?}$$



No!

Theorem (Gödel's First Incompleteness theorem)

Let A be any consistent and recursive set of $\mathcal{L}_{NT}$-formulas. Then, there is a sentence θ such that $\mathbb{N} \models \theta$ but $A \nvdash \theta$.

* a set of nonlogical axioms A in a language $\mathcal{L}$ is called complete if for every $\mathcal{L}$-sentence σ, $A \vdash \sigma$ or $A \vdash \neg\sigma$. So, Gödel's theorem is about the incompleteness of the axioms A, not the deductive system (which is not incomplete).

From here on, everything is about $\mathbb{N}$ and $\mathcal{L}_{NT}$.

4.2 Complexity of $\mathcal{L}_{NT}$-formulas

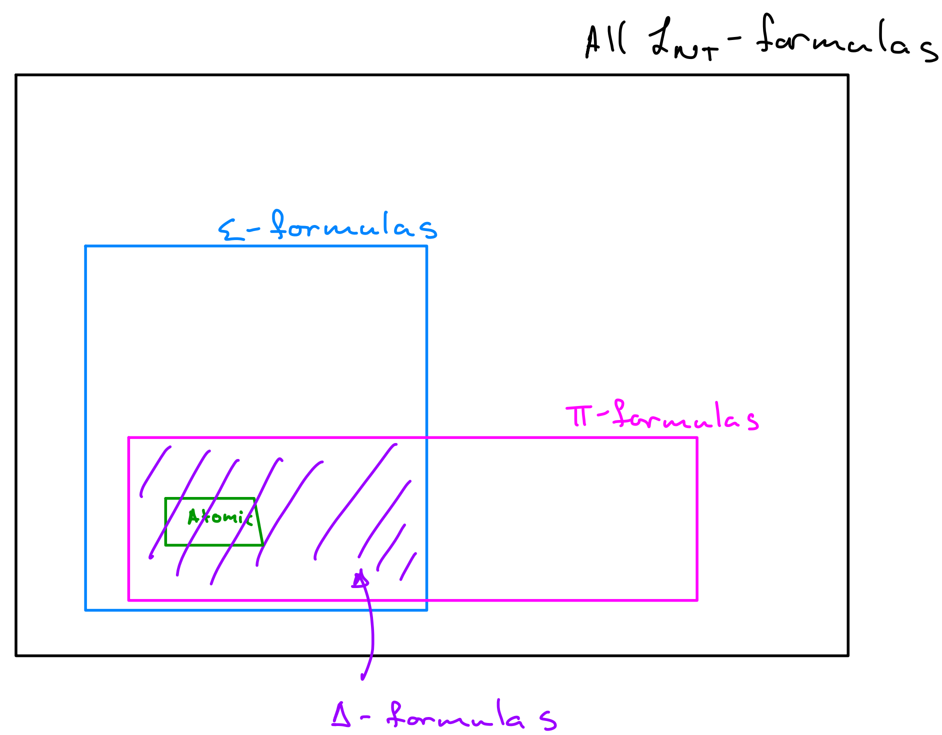
<u>Def</u> (Bounded Quantification) Suppose x does not occur in the term t. The <u>bounded quantifiers</u> are

$$(\forall x < t)\phi, \quad (\forall x \leq t)\phi, \quad \overbrace{(\exists x < t)}^{(\exists x)(x<t \wedge \phi)}\phi, \quad (\exists x \leq t)\phi$$

$(\forall x \leq t)\phi$ is shorthand for $\forall x[(x<t) \vee x=t) \rightarrow \phi]$

<u>Ex</u> $\underbrace{(\exists x < \bar{5})}_{\text{bounded}}(\underbrace{\forall y}_{\text{unbounded}}(x+y=y))$

We now define some special classes of formulas



change to Π

Def The Σ-formulas is the smallest set of formulas such that

1. it contains all atomic formulas and their negations
2. it is closed under $\vee$ and $\wedge$
   (e.g. α, β Σ-formulas implies $\alpha \vee \beta$ is a Σ-form.)
3. it is closed under all bounded quantifiers
   (e.g. α a Σ-formula implies $(\forall x < t)\alpha$ is a Σ-formula, where x is not in t)
4. it is closed under unbounded $\exists$. change to $\forall$
   (e.g. α a Σ-formula implies $(\exists x)\alpha$ is a Σ-formula.)

* for Π-formulas, (4) changes to unbounded $\forall$

* for Δ-formulas, (4) is removed, so no unbounded quantifiers.

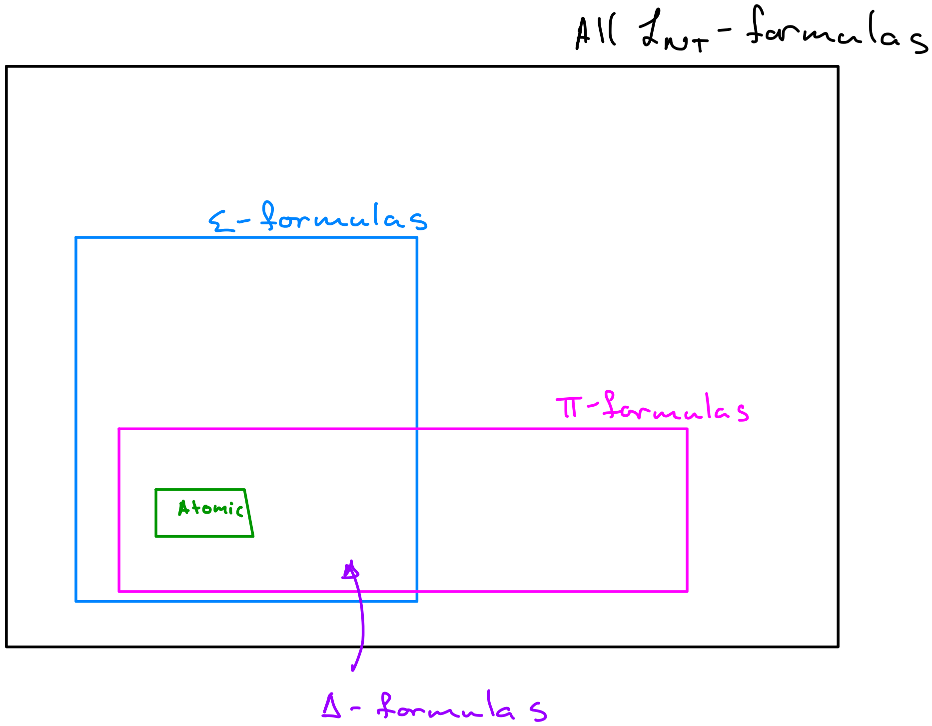
! we've seen that there are $\mathcal{L}_{NT}$-sentences ϕ s.t. $\mathbb{N} \models \phi$ but $N \nvdash \phi$ (e.g. $\phi :\equiv (\forall x)(\neg x < x)$)

However, we'll see that if ϕ is a Δ-formula then

$\mathbb{N} \models \phi$ implies $N \vdash \phi$ ← and this is true for Σ-formulas!

AND $\mathbb{N} \not\models \phi$ implies $N \vdash \neg\phi$

Ex Generate examples in each area:



Note: In Exercise 4.2.1, (b) is not a Σ-form., but it is logically equiv. to one.

Ex Let $\alpha :\equiv (\forall x < \overline{17})\underbrace{(\exists y)(x+y=\overline{17})}_{\beta}$

- α is a Σ-formula
- $\neg\alpha$ is not a Σ- or Π-formula. However, if we consider logical equivalence,

$$
\begin{aligned}
\neg\alpha &\equiv \neg(\forall x < \overline{17})\,\beta \\
&\equiv \neg(\forall x)(x > \overline{17} \vee \beta) \\
&\equiv (\exists x)(x < \overline{17} \wedge \neg\beta) \\
&\equiv \underbrace{(\exists x < \overline{17})((\forall y)(x+y \neq \overline{17})}_{\phi}
\end{aligned}
$$

Π-formula

Thus $\neg\alpha \models \phi$ and $\phi \models \neg\alpha$, so $\neg\alpha$ is logically equivalent to a Π-formula

<u>Lemma</u> If α is a Σ form. then $\neg\alpha$ is logically equivalent to a Π-formula, and vice versa.

* what does this mean for Δ-formulas?

* sometimes, Σ, Π, Δ-formulas are defined as anything logically equivalent to...