CHAPTER 5

SYNTACTIC INCOMPLETENESS
GROUNDWORK

5.3 Representable (and definable) sets and functions

why Σ-formulas and Δ-formulas.

First answer: we know that $N \vdash \phi \Rightarrow \mathbb{N} \models \phi$

but the converse is not true for all formulas ϕ.

However, it is true for Σ-sentences.

Prop. 5.3.13 Let ϕ be a Σ-sentence. If $\mathbb{N} \models \phi$, then $N \vdash \phi$.

Pf Jerilyn — thanks!

More is true for Δ-formulas.

Prop. 5.3.14 Let ϕ be a Δ-sentence.

- If $\mathbb{N} \models \phi$, then $N \vdash \phi$.
- If $\mathbb{N} \not\models \phi$, then $N \vdash \neg\phi$

Second answer: Representable sets can be defined by Σ-formulas.

what do these mean?

Representable and definable sets

Let $f(x) = 2^x$. This function is not in our language — can we still use it in sentences?

Yeah, kind of...

Let α be the formula

Remember: $\overline{2} = \underbrace{SS}_{2\text{-times}}0$

$$\alpha(x,y) :\equiv y = E(\overline{2}, x).$$

Notice that, for all $a \in \mathbb{N}$,

$$\mathbb{N} \models (\forall y)\big[\underbrace{\alpha(\overline{a}, y)}_{\mathbb{N} \text{ thinks } \alpha(a,y) \text{ is true}} \longleftrightarrow \underbrace{y = \overline{2^a}}_{\mathbb{N} \text{ thinks } y = 2^a}\big]$$

* this says that as far as $\mathbb{N}$ is concerned,

$$y = 2^x \quad \text{and} \quad \alpha(x,y)$$

are the same. We say α <u>defines</u> $f(x) = 2^x$.

In fact, looking back at Lem. 2.8.4, we find that

$$N \vdash (\forall y)\big[\alpha(\overline{a}, y) \longleftrightarrow y = \overline{2^a}\big]$$

* so as far as any model of N is concerned, by soundness

$$y = 2^x \quad \text{and} \quad \alpha(x,y)$$

are the same. We say α <u>represents</u> $f(x) = 2^x$.

<u>Def</u> If $A \subseteq \mathbb{N}^k$, we say

- A is <u>definable in $\mathbb{N}$</u>, if there is an $\mathcal{L}_{NT}$-formula $\varphi(x_1, \ldots, x_k)$ s.t. for all $a_1, \ldots, a_k \in \mathbb{N}$

$$(a_1, \ldots, a_k) \in A \text{ iff } \mathbb{N} \models \varphi(\overline{a_1}, \ldots, \overline{a_k}).$$

- A is weakly representable in N, if there is an $\mathcal{L}_{NT}$-form. $\varphi(x_1, \dots, x_k)$ s.t. $\forall a_1, \dots, a_k \in \mathbb{N}$

$$(a_1, \dots, a_k) \in A \text{ iff } N \vdash \varphi(\bar{a}_1, \dots, \bar{a}_k).$$

- A is representable in N, if

$$(a_1, \dots, a_k) \in A \implies N \vdash \varphi(\bar{a}_1, \dots, \bar{a}_k).$$

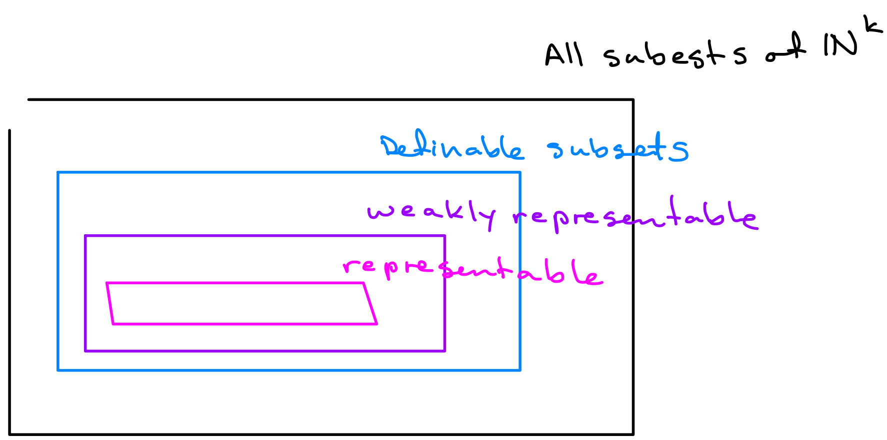
$$(a_1, \dots, a_k) \in A \implies N \vdash \neg \varphi(\bar{a}_1, \dots, \bar{a}_k).$$

* Remember, if $f : \mathbb{N} \to \mathbb{N}$ is a function, then

$$f(x) = y \text{ iff } (x, y) \in f.$$

So, f is definable if there is an $\mathcal{L}_{NT}$-formula $\varphi(x, y)$ s.t. $\forall a, b \in \mathbb{N}$

$$f(a) = b \text{ iff } \mathbb{N} \models \varphi(\bar{a}, \bar{b})$$



Proposition 5.3.6 A function is representable iff it is weakly representable.

Ex Show that the set of primes in $\mathbb{N}$ is definable with a Δ-formula.

Need a formula $Prime(x)$ s.t. $\forall a \in \mathbb{N}$

a is prime iff $\mathbb{N} \models Prime(\bar{a})$.

Take 1

$$Prime(x) :\equiv x > \bar{1} \wedge \forall y \left[((\exists z)(x = yz)) \rightarrow (y = \bar{1} \vee y = x) \right]$$

This works, but it's not a Δ-formula.

Take 2

$$Prime(x) :\equiv x > \bar{1} \wedge \forall y \leq x \left[((\exists z \leq x)(x = yz)) \rightarrow (y = \bar{1} \vee y = x) \right]$$

why take the extra time to find a Δ-formula?

Corollary 5.3.15 If $A \subseteq \mathbb{N}^k$ is definable by a Δ-formula, then A is representable.

Ex The set of primes of $\mathbb{N}$ is representable (since we showed it has a Δ-definition).

4.5, 5.5–5.8 Coding, Gödel numbering, and $\mathbb{N}$

Remember the goal...

<u>Theorem</u> (Gödel's First Incompleteness theorem)

Let A be any consistent and recursive set of $\mathcal{L}_{NT}$-formulas. Then, there is a sentence Θ such that $\mathbb{N} \models \Theta$ but $A \nvdash \Theta$.

The idea is to choose

$$\Theta :\equiv \text{“This sentence is not deducible from } A\text{”}$$

* so if Θ is deducible from A then $\mathbb{N} \models \Theta$ by soundness, but then ...?... contradiction... so Θ must not be deducible from A. Done.

* but anyway, how could we write such a sentence?

We will need to "code" $\mathcal{L}_{NT}$-formulas, and ultimately, deductions too.

Terms, Formulas, and Deductions	Actual natur numbers
ϕ $\longrightarrow$	$\ulcorner \phi \urcorner$
$D = (\phi_1, \ldots, \phi_k)$ $\longrightarrow$	$\ulcorner D \urcorner$

For example

$$0 = 0 \longrightarrow \ulcorner 0 = 0 \urcorner = 2^8\, 3^{1025}\, 5^{1025}$$

And we want to do this in a way that's "easy" to decode.

Coding strings of characters

STEP 1 Strings of characters $\longrightarrow$ strings of numbers

$\mathcal{L}_{NT}$	$\neg$	$\vee$	$\forall$	$=$	0	S	$+$	$\cdot$	E	$<$	$($	$)$	v_i
$\mathbb{N}$	1	3	5	7	9	11	13	15	17	19	21	23	$2i$

Ex

$$T = v_3 \longrightarrow = S0v_3 \longrightarrow (7, 11, 9, 6)$$

STEP 2 Strings of numbers $\longrightarrow$ single number

* remember we want to be able to decode.

Def Let $(a_1, \ldots, a_k) \in \mathbb{N}^k$. Define

$$\langle a_1, \ldots, a_k \rangle = \begin{cases} 1 & \text{if } k = 0 \\ \underline{p_1^{a_1+1}\, p_2^{a_2+1} \cdots p_k^{a_k+1}} & \text{if } k > 0 \end{cases}$$

p_i is the i^{th} prime

* add 1 · , exp nent so we "see" 1 prime for each a_i

Ex

$$(7, 11 \quad , 6) \longrightarrow \langle 7, 11, 9, 6 \rangle = 2^8\, 3^{12}\, 5^{10}\, 7^7$$

$$\shortparallel$$

1 094 161 288 657 500 000 000

Ex

$$0 = 0 \longrightarrow = 00 \longrightarrow (7, 9, 9) \longrightarrow \langle 7, 9, 9 \rangle = 2^9\, 3^{16}, 5^{10}$$

$$\shortparallel$$

255 245 000 000 000

$$2^9\, 3^{10}\, 5^{10} \neq \ulcorner 0 = 0 \urcorner$$

This allows us to code any sequence of charaters. However, we want to be more intentional when coding terms and formulas (and deductions).

<u>Def 5.7.1</u> (Gödel Numbering) Let s be any string of $\mathcal{L}_{NT}$-symbols. We (recursively) define $\ulcorner s \urcorner$ as follows:

$$\ulcorner s \urcorner \begin{cases} \langle 1, \ulcorner \alpha \urcorner \rangle & \text{if } s :\equiv (\neg \alpha) \text{ for } \alpha \text{ an } \mathcal{L}_{NT} \text{ form.} \\ \langle 3, \ulcorner \alpha \urcorner, \ulcorner \beta \urcorner \rangle & \text{if } s :\equiv (\alpha \vee \beta) \quad \text{..} \\ \langle 5, \ulcorner v_i \urcorner, \ulcorner \alpha \urcorner \rangle & \text{if } s :\equiv (\forall v_i)(\alpha) \\ \langle 7, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle & \text{if } s :\equiv = t_1 t_2 \text{ with } t_1, t_2 \text{ terms} \\ \langle 9 \rangle & \text{if } s :\equiv 0 \\ \langle 11, \ulcorner t \urcorner \rangle & \text{if } s :\equiv St \\ \langle 13, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner & \text{if } s :\equiv + t_1 t_2 \\ \langle 15, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle & s :\equiv \cdot t_1 t_2 \\ \langle 17, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle & s :\equiv E t_1 t_2 \\ \langle 19, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle & s :\equiv < t_1 t_2 \\ \langle 2i \rangle & s :\equiv v_i \\ 3 & \text{otherwise} \end{cases}$$

<u>Ex</u> compute

(a) $\ulcorner 0 = 0 \urcorner$ (b) $\ulcorner \overline{1} = v_3 \urcorner$ (c) $\ulcorner < 0 + \urcorner$

(a) $0 = 0 \longrightarrow = 0\,0 \longrightarrow \langle 7, \ulcorner 0 \urcorner, \ulcorner 0 \urcorner \rangle = \langle 7, \langle 9 \rangle, \langle 9 \rangle \rangle$

$= \langle 7, 2^{10}, 2^{10} \rangle = 2^{8} 3^{2^{10}+1} 5^{2^{10}+1} = \boxed{2^{8} \cdot 3^{1025} \cdot 5^{1025}}$

(b) $\ulcorner T = v_3 \urcorner$

$T = v_3 \to S0 = v_3 \to = S0v_3 \to \langle 7, \ulcorner S0 \urcorner, \ulcorner v_3 \urcorner \rangle$

$= \langle 7, \langle 11, \ulcorner 0 \urcorner \rangle, \langle 6 \rangle \rangle = \langle 7, \langle 11, \langle 9 \rangle \rangle, \langle 6 \rangle \rangle$

$= \langle 7, \langle 11, 2^{10} \rangle, 2^7 \rangle = \langle 7, 2^{12} 3^{2^{10}+1}, 2^7 \rangle$

$= \boxed{2^8 \cdot 3^{2^{12} 3^{2^{10}+1}+1} \cdot 5^{2^7+1}}$

(c) $\ulcorner < 0 + \urcorner = 3$

<u>Ex</u> suppose ϕ is a formula such that

$$\ulcorner \phi \urcorner = 2^{20} \cdot 3^{2^{10}+1} \cdot 5^{2^{12} 3^{2^{12} 3^{2^{10}+1}+1}+1}$$

Decode to find ϕ.

$$\ulcorner \phi \urcorner = \langle 19, 2^{10}, 2^{12} 3^{2^{12} 3^{2^{10}+1}+1} \rangle$$

$$= \langle 19, \langle 9 \rangle, \langle 11, 2^{12} 3^{2^{10}+1} \rangle \rangle$$

$$= \langle 19, \langle 9 \rangle, \langle 11, \langle 11, 2^{10} \rangle \rangle \rangle$$

$$= \langle 19, \langle 9 \rangle, \langle 11, \langle 11, \langle 9 \rangle \rangle \rangle \rangle$$

$$= \langle 19, \ulcorner 0 \urcorner, \langle 11, \langle 11, \ulcorner 0 \urcorner \rangle \rangle \rangle$$

$$= \langle 19, \ulcorner 0 \urcorner, \langle 11, \ulcorner S0 \urcorner \rangle \rangle$$

$$= \langle 19, \ulcorner 0 \urcorner, \ulcorner SS0 \urcorner \rangle$$

$$= \ulcorner < 0 SS0 \urcorner$$

so $\boxed{\phi :\equiv 0 < \bar{2}}$

Some important sets

Let VARIABLE $= \{a \in \mathbb{N} \mid a = \ulcorner v \urcorner$ for some variable $v\}$

Let TERM $= \{a \in \mathbb{N} \mid a = \ulcorner t \urcorner$ for some $\mathcal{L}_{NT}$-term $t\}$

Let FORMULA $= \{a \in \mathbb{N} \mid a = \ulcorner \phi \urcorner$ " " " formula $\phi\}$

Proposition All three of the above have a Δ-definition — thus they are representable.

pf for VARIABLE

consider the formula

$$V_1(x) :\equiv \text{“} x = 2^{y+1} \text{ for some even } y > 0 \text{”}$$

$$\equiv \exists y \left((\exists z)(y = z \cdot z) \wedge y > 0 \wedge x = E(2, Sy) \right)$$

Then, $\forall a \in \mathbb{N}$,

$$a \in \text{VARIABLE iff } \mathbb{N} \models V_1(\bar{a})$$

so V_1 defines VARIABLE, but it's not a Δ definition. Try again...

$$V_2(x) :\equiv \exists (y < x) \left((\exists z)(y = z \cdot z) \wedge y > 0 \wedge x = E(2, Sy) \right)$$

$\square$

Remember, we want to be able to formalize

$\theta :\equiv$ " This sentence is not deducible from A"

Coding Deductions

we need

① to code a sequence of formulas $D = (\phi_1, \dots, \phi_k)$

② to know which sequences are actually deductions

Regarding ① ...

$$\ulcorner D \urcorner = 5^{\ulcorner \phi_1 \urcorner} \cdot 7^{\ulcorner \phi_2 \urcorner} \cdots p_{k+2}^{\ulcorner \phi_k \urcorner}$$

* note that Gödel numbers of terms and formulas are all even

* numbers whose smallest prime factor is 3 are garbage

* so, if the smallest prime factor is 5, then the number represents a sequence

Regarding ② ...

Proposition If A is a representable set of axioms, then

$$\text{DEDUCTION}_A = \left\{ (c, f) \;\middle|\; \begin{array}{l} c \text{ is the code for a} \\ \text{deduction from } A \\ \text{of a formula with} \\ \text{Gödel number } f \end{array} \right\}$$

is representable.

CHAPTER 6 INCOMPLETENESS THEOREMS

The Goal

<u>Gödel's 1st Incompleteness Thm</u> If A is <u>any</u> consistent and recursive set of $\mathcal{L}_{NT}$-formulas, then there is a sentence θ s.t. $\mathbb{N} \models \theta$ but $A \nvdash \theta$.

The Idea

$\theta :\equiv$ "This sentence is not deducible from A"

The Setup

Let A be any consistent set of $\mathcal{L}_{NT}$-formulas.

<u>Def</u>

$$\text{AxiomOfA} = \{\ulcorner \alpha \urcorner \mid \alpha \in A\} \subseteq \mathbb{N}$$

$$\text{Thm}_A = \{\ulcorner \phi \urcorner \mid A \vdash \phi\} \subseteq \mathbb{N}$$

<u>Def</u> A is <u>recursive</u> if AxiomOfA is representable.

<u>Lemma 6.35</u> If A is recursive, then Thm_A is definable by a Σ-formula which we call $\text{thm}_A(v_1)$. That is,

$$n \in \text{Thm}_A \text{ iff } \mathbb{N} \models \text{thm}_A(\bar{n})$$

! This does not say Thm_A is <u>representable</u> (which it would be if there were a Δ-definition).

The proof of 6.3.5 makes use of our work in previous chapter, together with the following Lemma.

<u>Lemma 6.3.3</u> If $A \subseteq \mathbb{N}$ is representable, then it has a Σ-definition.

<u>Lemma 6.2.2. (Gödel's Self Reference Lemma)</u>

Let $\varphi(v_1)$ be any $\mathcal{L}_{NT}$-formula (with only v_1 free). Then there is a sentence θ s.t.

$$N \vdash (\theta \leftrightarrow \varphi(\ulcorner\theta\urcorner))$$

* Thus, θ says "φ is true about me".

! Applying this when φ is $\neg thm_A(v_1)$, we have

$$N \vdash (\theta \leftrightarrow \neg tm_A(\ulcorner\theta\urcorner))$$

so here, θ says "I am not a theorem deducible from A"

<u>Theorem 6.3.6. Gödel's First Incompleteness Theorem</u>

If A is <u>any</u> consistent and recursive set of $\mathcal{L}_{NT}$-formulas, then there is a sentence θ s.t. $\mathbb{N} \models \theta$ but $A \nvdash \theta$.

<u>pf</u> we may assume $A \vdash N$ (o/w we are done).

consider $THM_A = \{\ulcorner\phi\urcorner \mid A \vdash \phi\}$. By 6.3.5,

there exists a Σ-formula $thm_A(v_1)$ s.t.

$$n \in THM_A \quad \text{iff} \quad \mathbb{N} \models thm_A(\bar{n}).$$

Applying the self-Ref. Lemma to $\neg thm_A(v_1)$,

there exists θ s.t.

$$N \vdash (\theta \longleftrightarrow \neg thm_A(\overline{\ulcorner\theta\urcorner})).$$

By Soundness,

$$N \models (\theta \longleftrightarrow \neg thm_A(\overline{\ulcorner\theta\urcorner}));$$

so

$$\begin{aligned} \mathbb{N} \models \theta \quad &\text{iff} \quad \mathbb{N} \not\models thm_A(\overline{\ulcorner\theta\urcorner}) \\ &\text{iff} \quad \ulcorner\theta\urcorner \notin THM_A \\ &\text{iff} \quad A \nvdash \theta. \end{aligned}$$

thm_A defines THM_A

If $\mathbb{N} \models \theta$ then $A \nvdash \theta$ and we are done. Thus towards a contradiction, assume $\mathbb{N} \not\models \theta$ and $A \vdash \theta$.

Then $\mathbb{N} \models thm_A(\overline{\ulcorner\theta\urcorner})$. Remember, $thm_A(\overline{\ulcorner\theta\urcorner})$ is a Σ-sententence, so by Prop 5.3.13 $N \vdash thm_A(\overline{\ulcorner\theta\urcorner})$!! Then, by choice of θ, $N \vdash \neg\theta$, and as $A \vdash N$, $A \vdash \neg\theta$. Thus $A \vdash \theta \wedge \neg\theta$, which contradicts the fact that A is consistent. $\square$