EXAM 1 - REVIEW QUESTIONS

LINEAR ALGEBRA

Questions (answers are below)

Examples. For each of the following, either provide a specific example which satisfies the given description, or if no such example exists, briefly explain why not.

(1) (JW) A skew-symmetric matrix A such that the trace of A is 1
(2) (MH) A nonzero, square matrix A that is both a scalar matrix and a skew-symmetric matrix.
(3) (OS) A nonsingular matrix A such that its transpose is row equivalent to the transpose of the inverse of A.
(4) (RYT) **[Not on this exam.]** A one-to-one and onto matrix transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(\mathbf{x}) = A\mathbf{x}$ for a non-invertible matrix $A \in M_{n\times n}$.
(5) (JG) A diagonal matrix whose trace is equal to that of a skew symmetric matrix.
(6) (CH) **[Not on this exam.]** A 3x3 matrix whose determinant is 4.
(7) (HR) An upper triangular matrix which is skew symmetric
(8) (SC) Two matrices A and B with no zero entries, for which $\mathrm{Tr}(A + B) = 0$.
(9) (MH) An invertible matrix A which has a trace of 0
(10) (JS) 2x2 matrix A where $A^T = A^{-1}$.
(11) (MJ) An augmented matrix that corresponds to a linear system with infinitely many solutions.
(12) (RJT) A skew symmetric matrix A for which the homogeneous system $Ax = 0$ has a nontrivial solution.
(13) (JAW) **[Not on this exam.]** A 3×3 symmetric matrix whose determinant is 0.
(14) (JRS) A singular matrix A and a nonsingular matrix B such that $AB = I_n$.
(15) (RS) An elementary matrix obtained by performing all three types of row operations on I_n.
(16) (TD) **[Not on this exam.]** A permutation $\sigma \in S_5$ with exactly two inversions.
(17) (KG) A matrix that is the inverse of

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

(18) (GD)**[Not on this exam.]** Two 2×2 matrices A and B where B is the inverse of A and A has determinant 0.
(19) (QH) Two elementary matrices whose product is the 2×2 identity matrix.
(20) (JF) A 3×3 matrix A, where $A\mathbf{x}$=0 has no nontrivial solution.
(21) (BS) A matrix A such that the matrix transformation $f : R^2 \to R^2$ defined by $f(u) = Au$ is a reflection across the x-axis, if $u = \begin{bmatrix} x \\ y \end{bmatrix}$?
(22) (AV) Three matrices $A, B,$ and C exist such that $AB = AC$, but $B \neq C$.
(23) (AR) A 2×2 matrix A such that all entries of A are equal and A has an inverse.
(24) (MRR) A nonsingular $n \times n$ matrix A, such that $Ax = b$ has infinitely many solutions for every $n \times 1$ matrix b.

True or False.

(a) (JW) There exist an $A \in M_{n\times n}$, such that $A \neq 0$, $A \neq I_n$, and $AB = BA$ for all $B \in M_{n\times n}$.
(b) (MH) The reduced row echelon form (RREF) of a singular matrix has a row of zeroes.
(c) (OS) $\neg\exists x \in \mathbb{R}(p(x) \Rightarrow q(x)) \equiv \forall x \in \mathbb{R}(p(x) \wedge \neg q(x))$.
(d) (RYT) There exists an $n \times n$ matrix $A = [a_{ij}]$ such that $a_{ij} = (i - j)^3$ and $A + A^T$ is invertible.

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(e) (JG) If you can row-reduce an $n \times n$ matrix, A, to the identity matrix, then the system, $Ax = b$, cannot be inconsistent.

(f) (MJ) A homogeneous system of equations can be inconsistent.

(g) (CH) $\exists X(P(X) \wedge Q(X)) \equiv (\exists X P(X) \wedge (\exists X Q(X))$

(h) (HR) The $n \times n$ zero matrix is upper triangular.

(i) (SC) $\exists A \in M_{n\times n}(\forall B \in M_{n\times n}$ such that $AB = BA = I_n$ and $A \neq I_n)$

(j) (MH) Every upper triangular matrix is invertible.

(k) (JS) If A and B are both nonsingular $n \times n$ matrices, then AB is nonsingular and $(AB)^{-1} = A^{-1}B^{-1}$

(l) (RS) **[Not on this exam.]** The determinant of a skew-symmetric matrix is always zero.

(m) (JRS) If AB is a singular $n \times n$ matrix, then A is a singular $n \times n$ or B is a singular $n \times n$ matrix.

(n) (RJT) There exists some 3×3 skew symmetric matrix that is row equivalent to the identity matrix.

(o) (TD) A singular matrix A is row equivalent to I_n.

(p) (JAW) **[Not on this exam.]** All sub matrices of any skew symmetric matrices have a 0 entry and are upper or lower triangular.

(q) (KG) If an $n \times n$ matrix A is singular then $Ax = 0$ has only the trivial solution.

(r) (GD) If the RREF of an $n \times n$ matrix A has a row of zeroes, A is singular.

(s) (QH) An elementary matrix is nonsingular.

(t) (JF) Any matrix equivalent to an identity matrix is nonsigular.

(u) (BS) The row echelon form of a matrix is unique.

(v) (AV) The sum of two $n \times n$ upper (or lower) triangular matrices is upper (or lower) triangular.

(w) (AR) If AB is nonsingular, then A and B must both be nonsingular.

(x) (MRR) **[Not on this exam.]** When using the definition of the determinant to compute the determinant of an upper triangular matrix with nonzero entries on the main diagonal, the sum can contain multiple nonzero terms.

Answers

Examples - Answers.

(1) (JW) Not possible. Skew-symmetric matrices have all zeros on the main diagonal, so the trace must be zero.

(2) (MH) Not possible. A skew-symmetric matrix must have all zeroes on the main diagonal, so it can't be a scalar matrix unless it's the zero matrix.

(3) (OS) Possible. Consider $\begin{bmatrix} -4 & 3 \\ 2 & -3 \end{bmatrix}$.

(4) (RYT) Impossible. Since A is non-invertible, there exists a $v \in \mathbb{R}^n$, such that $A\mathbf{x} = \mathbf{v}$ does not have a unique solution. Thus, no such f exists.

(5) (JG) Possible. Consider $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{bmatrix}$.

(6) (CH) Possible. Consider $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

(7) (HR) Possible. Consider a zero matrix.

(8) (SC) Consider 43(a) from Chapter 1.3; $\operatorname{Tr}(A+B) = \operatorname{Tr}(A) + \operatorname{Tr}(B)$. Any matrices A and B for which $\operatorname{Tr}(A) = -\operatorname{Tr}(B)$ will have a zero trace. For example $A = \begin{bmatrix} 2 & 1 \\ 7 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & -\pi \\ 11 & -6 \end{bmatrix}$

(9) (MH) Possible. Consider $\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$.

(10) (JS) Possible. Consider $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(11) (MJ) Possible. Consider $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$. But, the following is NOT an example $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

(12) (RJT) Possible. Consider $\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$.

(13) (JAW) Possible. Consider $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

(14) (JRS) Trick question. Not possible. A is singular, so by definition, there exists no B such that $AB = BA = I_n$

(15) (RS) Not possible. By definition, an elementary matrix is a matrix obtained by performing a **single** elementary row (or column) operation on the identity matrix. For instance, $\begin{bmatrix} 0 & 5 & 0 \\ 0 & 0 & 1 \\ 3.5 & 0 & 0 \end{bmatrix}$ is not an elementary matrix.

(16) (TD) Possible. Consider $\sigma = 13425$.

(17) (KG)

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

(18) (GD) Not possible. If an $n \times n$ matrix A has a determinate of 0, the matrix is singular and has no inverse B such that $AB = I_n$.

(19) (QH) Possible. The product of $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(20) (JH) Possible. Consider the identity matrix, or any matrices that are row equaivalent to the identity matrix.

(21) (BS) Possible. Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

(22) (AV) Possible. Consider $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, and $C = \begin{bmatrix} -2 & 7 \\ 5 & -1 \end{bmatrix}$.

(23) (AR) Not possible. Such a matrix will be row equivalent to a matrix with a row of zeros and, consequently, will not be row equivalent to I_2. By the Invertibility Theorem, such a matrix does not have an inverse.

(24) (MRR) Not possible. By the Invertibility Theorem, if A is nonsingular, then $Ax = b$ has a unique solution.

True or False - Answers.

(a) (JW) T, consider $A = 2 \cdot I_n$.

(b) (MH) T, if the matrix is non-singular, then its reduced row echelon form is the identity matrix I_n but since it's singular, that means the RREF can't be the identity matrix. Thus, there must be at least one row of zeroes. We know from the definition of RREF that if there are any zero rows, it must appear at the bottom of the matrix.

(c) (OS) T, consider negation of quantifiers and implications.

(d) (RYT) F, note A is a skew-symmetric matrix, which implies $A + A^T = \mathbf{0}$, which is not invertible.

(e) (JG) T, by the Invertibility Theorem

(f) (MJ) F, the zero vector will always be a solution and therefore the system will not be inconsistent.

(g) (CH) F, consider when $P(X)$ is "X an odd number" and $Q(X)$ is "X is an even number."

(h) (HR) T, A zero square matrix is lower triangular, upper triangular, and diagonal.

(i) (SC) F, the proposition states that there exists one matrix which is the inverse of every $n \times n$ matrix, which is untrue.

(j) (MH) F, For an upper (or lower) triangular matrix to be invertible, all values along the main diagonal must equal nonzero numbers. There is no distinction in the definition of a triangular matrix that the values of the main diagonal all need to be nonzero.

(k) (JS) F, in general the formula is $(AB)^{-1} = B^{-1}A^{-1}$.

(l) (RS) F, consider a 2×2 skew symmetric matrix such as $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

(m) (JRS) T, contrapositive of Theorem 1.6. If A and B are both nonsingular $n \times n$ matrices, then AB is nonsingular and $AB^{-1} = A^{-1}B^{-1}$.

(n) (RJT) F, We can write the general matrix as $\begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix}$, which has the REF form $\begin{bmatrix} 1 & -c/b & 0 \\ 0 & 1 & -b/a \\ 0 & 0 & 0 \end{bmatrix}$. Since there is a row of zeros the matrix is not row equivalent to the identity matrix.

(o) (TD) F. A singular matrix is not invertible and therefore not row equivalent to I_n. Only nonsingular matrices can be transformed into I_n.

(p) (JAW) F, while all submatrices of a skew symmetric matrix contain 0, they are not all upper or lower triangular.

(q) (KG) F, The homogeneous system $Ax = 0$ has a nontrivial solution if and only if A is singular. (That is, the reduced row echelon form of $A \neq I_n$.)

(r) (GD) T, by the Invertibility Theorem

(s) (QH) True. We can prove that by considering each type of elementary matrix.

(t) (JH) True. According to invertibility theorem.

(u) (BS) False. It is true that every $m \times n$ matrix is row equivalent to a unique RREF matrix, but most matrices are equivalent to many matrices in REF.

(v) (AV) True. Adding two matrices will change the entries unless you are adding two zeros as would be the case for two upper or lower triangular matrices.

(w) (AR) True. Since $P(AB) = I_n$, $(PA)B = I_n$, , using associativity of matrix multiplication. As we can see, $B^{-1} = (PA)$ and $A^{-1} = (PB)$, so the statement must be true. (See the last writing assignment.)

(x) (MRR) False, there would be only one nonzero term.