Chapter 1

(and some review)

Terminology (R.1)

Special sets of real numbers

positive & negative whole numbers and 0

(1) Integers (denoted $\mathbb{Z}$): $\ldots, -2, -1, 0, 1, 2, 3, \ldots$

(2) Rational Numbers (denoted $\mathbb{Q}$): numbers than are able to be written as $\frac{a}{b}$ with a, b both integers and $b \neq 0$.

(3) Irrational Numbers: real numbers that are <u>not</u> irrational

Ex Determine if each of the following are integers, rational numbers, or irrational numbers

(a) $\frac{1}{2}$ rational

(b) 17 integer, rational

(c) $\frac{\pi}{3}$ irrational

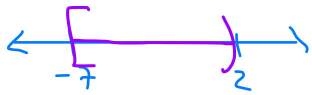
$\frac{2}{3}$

(d) $0.\overline{6}$ rational

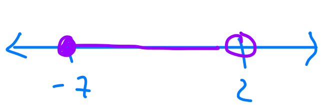
Inequalities and interval notation

Ex Describe graphically and in symbols

"all real number between -7 and 2, including -7 but not 2"

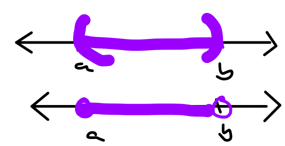
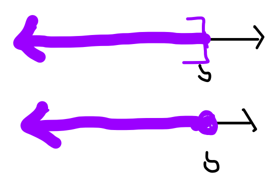


$-7 \le x < 2$



$[-7, 2)$

Some other examples

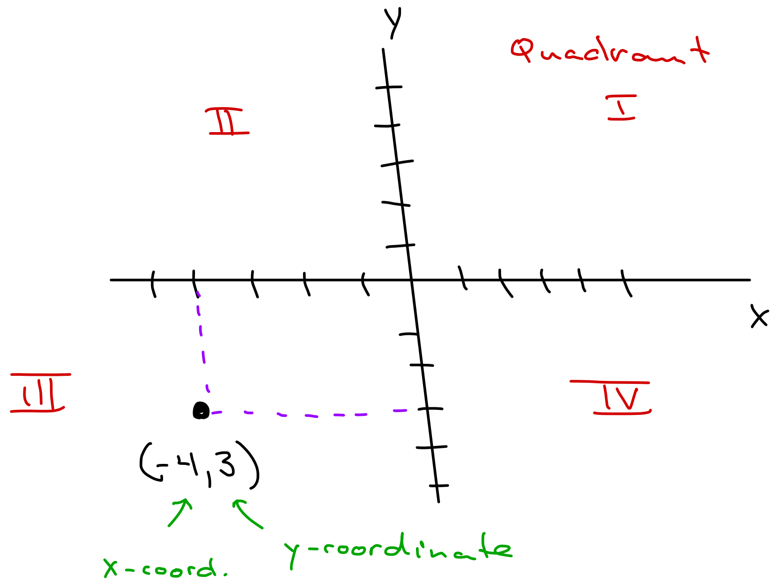
Inequality	Interval Notation	Graphically
$a < x < b$	(a, b)	
$x \le b$	$(-\infty, b]$	

* See book for others (pg. 3)

Rectangular Coordinate System

1.1

Plotting Points



Q: how do you know if (-4,3) is a point or an interval?

A: context

Q: how do you know if a point is on the graph?

Graphing Equations by Plotting Points

Ex Graph the equation $(y-1)^2 = x+1$ by plotting points.

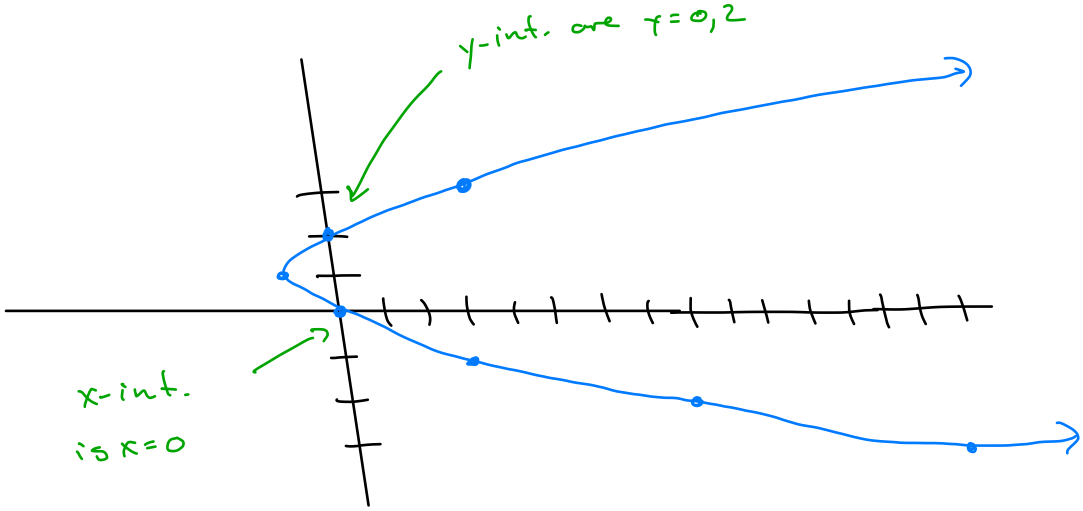
① If easy, solve for y or x first

$$x = (y-1)^2 - 1$$

② Plug in for one variable and solve for other

x	y
3	3
0	2
-1	1
0	0
3	-1
8	-2
15	-3

③ Plot the points and try to connect



The easiest points to find on a graph are the x any y-intercepts

Def

when $y=0$ → • An x-intercept of a graph is where it crosses the x-axis

when $x=0$ → • A y-intercept " " " " " the y-axis

Ex Find and plot the x- and y-intercepts of

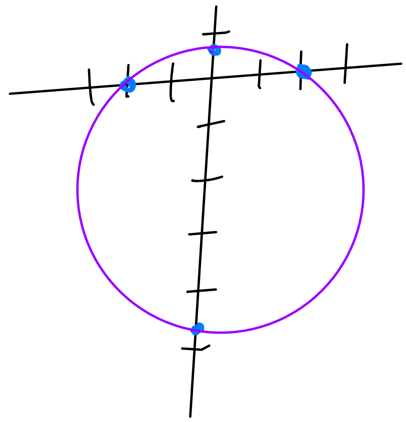
$$x^2+y^2+4y=4$$

x-intercepts: $y=0$ $x^2=4 \Rightarrow x=\pm 2$

y-intercepts: $x=0$ $y^2+4y-4=0 \Rightarrow y=\dfrac{-4\pm\sqrt{16+16}}{2}$

$\Rightarrow y=\dfrac{-4\pm\sqrt{32}}{2}$

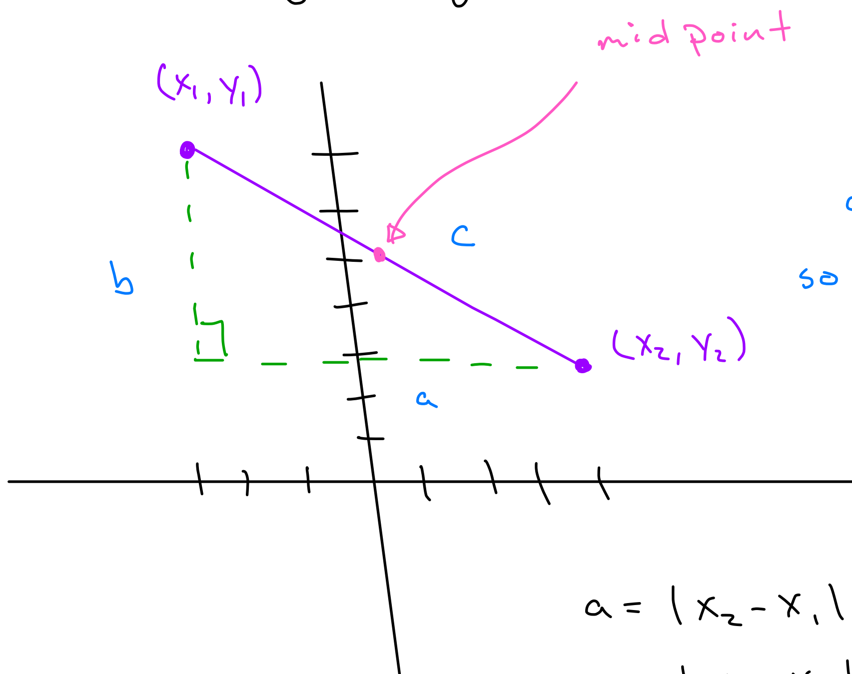
$\Rightarrow y \approx 0.828,\ -4.828$



it's a circle ... but may need more point to see

Distance and Midpoints

One often wants to find the distance b/w two points or the midpoint of the line joining them.



$a^2 + b^2 = c^2$

so $c = \sqrt{a^2+b^2}$

$a = |x_2 - x_1|$

$b = |y_2 - x_1|$

Theorem The distance b/c (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Returning to the picture, let's find the mid point. We see that...

x-value of midpoint: average of x-values

y-value of midpoint: average of y-values

Theorem The midpoint of the segment connecting (x_1, y_1) to (x_2, y_2) is

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

Ex Consider $P = (-3, 7)$, $Q = (4, 3)$. Find the distance b/w P & Q and midpoint of the segment joining them.

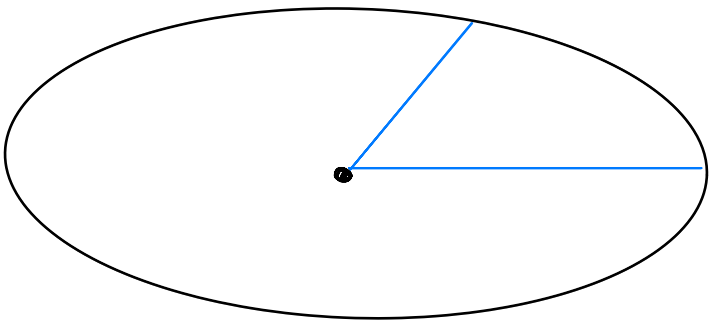
distance: $d = \sqrt{(4 - (-3))^2 + (3 - 7)^2}$
$= \sqrt{49 + 16} = \boxed{\sqrt{65}} \approx 8.06$

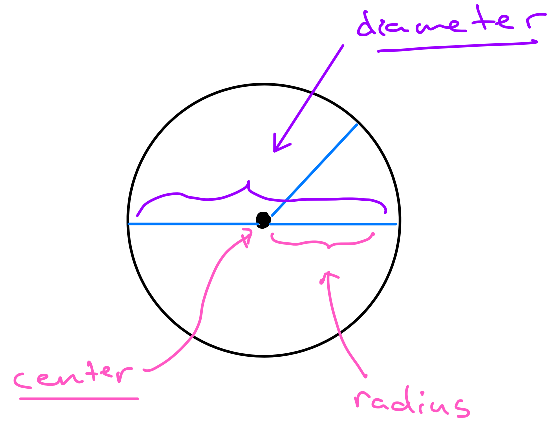
midpoint: $\left(\frac{-3 + 4}{2}, \frac{7 + 3}{2} \right) = \boxed{\left(\frac{1}{2}, 5 \right)}$

Circles $\boxed{1.2}$

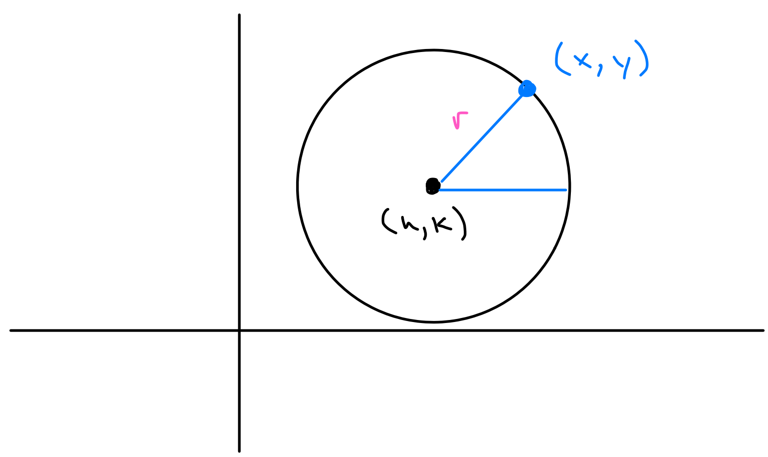
Q: what is a circle, in words?

Q: how can we find an equation for a circle?





<u>Def</u> A <u>circle</u> is the set of all points that are the same distance, called the <u>radius</u>, from a fixed point, called the <u>center</u>.



so

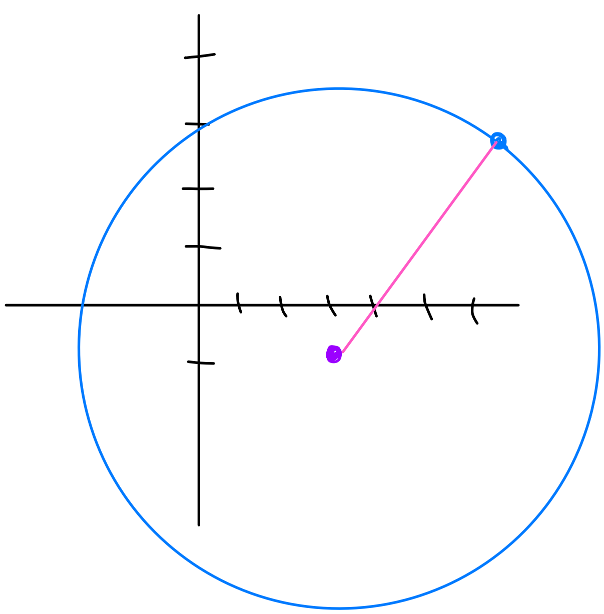
(x, y) is on the circle $\iff$ distance from (h, k) to (x, y) is r $\iff$ $\sqrt{(x-h)^2+(y-k)^2}=r$

Standard form of a circle

If a circle has its center at the point (h,k) and has radius r, then the circle is described by the equation

$$(x-h)^2 + (y-k)^2 = r^2 \qquad (\text{with } r>0)$$

Ex Let C be the circle centered at $(3,-1)$ that also passes through the point $(6,3)$. Write an equation for C and determine if $(2,5)$ is also on C



Need

- center : $(3,-1)$
- radius : ??

r must be the distance from $(3,-1)$ to $(6,3)$ so

$$r = \sqrt{(3-6)^2 + (-1-3)^2} = \sqrt{25} = 5$$

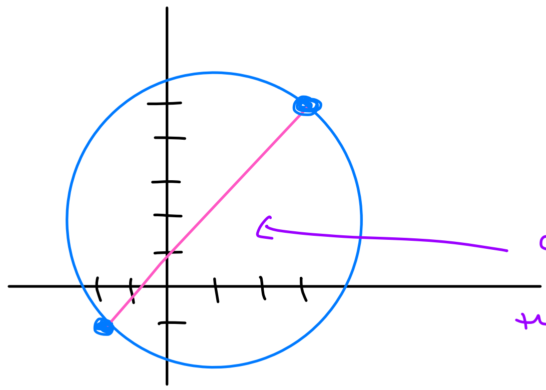
Standard equation

$$\boxed{(x-3)^2 + (y+1)^2 = 25}$$

Also $(2-3)^2 + (5+1)^2 = 1+36 = 37 \neq 25$

so $\boxed{\text{No} \quad (2,5) \text{ is not on the circle}}$

Ex Find an equation of a circle that has a diameter with endpoints $(-2,-1)$ and $(3,5)$.



Need

- center: $\left(\frac{-2+3}{2}, \frac{-1+5}{2}\right) = \left(\frac{1}{2}, 2\right)$
- radius: ??

radius must be distance from $\left(\frac{1}{2}, 2\right)$ to $(3,5)$

$$r = \sqrt{\left(\tfrac{1}{2}-3\right)^2 + (2-5)^2}$$
$$= \sqrt{(2.5)^2 + 3^2} = \sqrt{15.25}$$

standard eqn

$$\boxed{\left(x-\tfrac{1}{2}\right)^2 + (y-2)^2 = 15.25}$$

Ex Determine which of the following describe a circle. For those that do, find the center and radius.

(a) $(x-1)^2 + y - 2 = 9$ $\boxed{\text{No}}$ y only to first power. it is a parabola

(b) $x^2 + y^2 + 2y + 1 = 0 \Rightarrow x^2 + (y+1)^2 = 0$

$\boxed{\text{No}}$ does not have a positive radius

(c) $x^2 + 6x + y^2 = 2 \Rightarrow x^2 + 6x + \left(\frac{6}{2}\right)^2 + y^2 = 2 + \left(\frac{6}{2}\right)^2$

$\Rightarrow x^2 + 6x + 9 + y^2 = 11$

$\Rightarrow (x+3)^2 + y^2 = 11$ $\boxed{\text{Yes}}$

complete square: divide by 2 and square

center: $(-3,0)$

radius: $\sqrt{11}$

Functions & Relations

Relations encode how two quantities or variables are related.

This table gives the hours worked and money earned by 6 people

call this x → | ← call this y

Hours worked	Money Earned
10	160
5	45
7	70
10	90
20	300
6	30

<u>Def</u> A <u>relation</u> is any collection of ordered pairs.

<u>Ex</u>

think (x,y)

(a) $\{(10,160), (5,45), (7,70), (10,90), (20,300), (6,30)\}$

is a relation

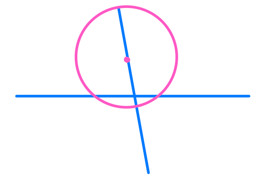
(b) $\{(\frac{1}{2},\pi), (\frac{1}{3},\pi), (\frac{1}{5},\pi)\}$

is a relation

(c) The set of all points satisfying

$$x^2 + (y-2)^2 = 9$$

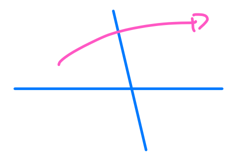
is a relation



(d) The set of all points satisfying

$$y = \sqrt{x+5} + 2$$

is a relation



<u>Def</u>

- The collection of all x-values appearing in a relation is called the <u>domain</u>.
- The collection of all y-values appearing in a relation is the <u>range</u>

<u>Ex</u> Find the domain and range of each relation above

(a) Domain: $\{5, 6, 7, 10, 20\}$
Range: $\{30, 45, 70, 90, 160, 300\}$

(b) Domain: $\{1/2, 1/3, 1/5\}$
Range: $\{\pi\}$

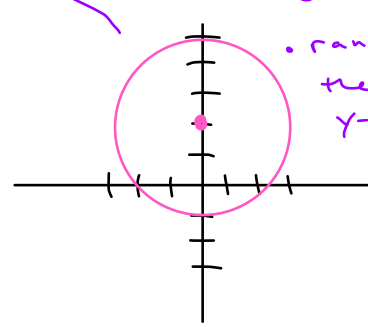
(c) Domain: $[-3, 3]$
Range: $[-1, 5]$

Let's do this graphically...

center is $(0, 2)$

radius is 3

- domain is set of x-values on circle
- range is the y-values



(d) Domain: $[-5, \infty)$

Range: $[2, \infty)$

Let's do this algebraically...

- $y = \sqrt{x+5} + 2 \implies x+5 \geq 0 \implies x \geq -5$

- $\sqrt{x+5} \geq 0$ so $\sqrt{x+5} + 2 \geq 2$

Functions

Functions model situations where there are inputs and corresponding outputs. In this case, an input should only yield one output.

Def A relation defines y as a function of x if for every x in the domain, there is only one corresponding value of y in the range.

Ex Determine which of the four relations in the first example define y as a function of x.

(a) Not a function

the x-value 10 corresponds to 2 y-values: 160, 90

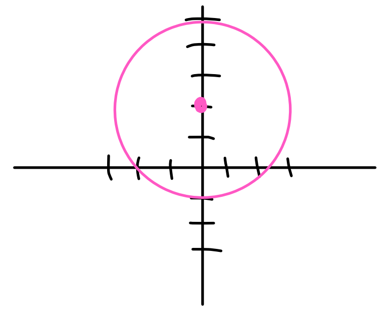
(b) Yes, it's a function

(c) No, it's not a function

Remember the graph:

center is $(0,2)$

radius is 3

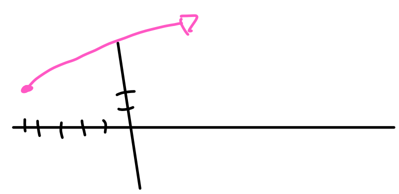


So, when $x=0$, y can be -1 or 5

(d) Yes, it's a function

algebraically: if you plug in x, you get only one y-value

this can be seen graphically too



Thinking graphically we get the following test to determine if a function is a relation or not.

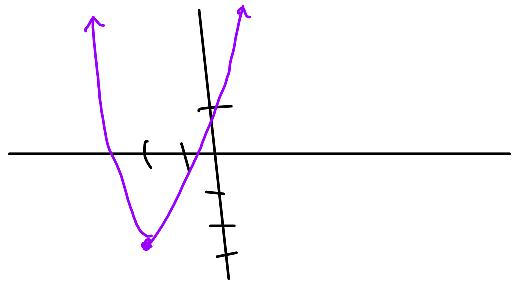
Vertical line test

A relation is a function if its graph does not intersect any vertical line in more than one point.

Function notation

Notice that $y = x^2 + 4x + 1$ defines y as a function of x, b/c every x yields only one y-value.

Or graphically,



In this case, we often write

$$y = f(x) \quad (\text{or } g(x), r(x), \ldots)$$

Here $f(x)$ means "the y-value associated to x"

For this problem, $f(x) = x^2 + 4x + 1$, so

$$f(0) = (0)^2 + 4(0) + 1 = 1$$

$$f(-2) = (-2)^2 + 4(-2) + 1 = -3$$

$$f(a) = a^2 + 4a + 1$$

$$f(a+h) = (a+h)^2 + 4(a+h) + 1$$

Note that

$f(a)$ does not mean "f times a" !! why not? ... context.

We can often talk about the concepts from before in this new notation: intercepts, domain, range.

OPTIONAL

Ex Find the x and y-intercepts of

$$f(x) = x^2 + 4x + 1$$

- x-int. when $y = 0$ (and $y = f(x)$)

solve $0 = x^2 + 4x + 1$

$$x = \frac{-4 \pm \sqrt{16-4}}{2} = \boxed{\frac{-4+\sqrt{12}}{2}, \frac{-4-\sqrt{12}}{2}}$$

≈ -0.268 ≈ -3.732

- y-int. when $x = 0$

$$y = f(0) = 0^2 + 4(0) + 1 = 1 \qquad \boxed{y = 1}$$

Ex Find the domain of each of the following.

(a) $h(x) = \dfrac{3-4x}{x^2-9}$

(b) $f(t) = \dfrac{1+\sqrt{3-t}}{t^2+1}$

(a) denominator must not be 0

$0 = x^2 - 9 = (x-3)(x+3) \implies x - 3 = 0 \quad x + 3 = 0$

Domain: $\boxed{\text{all } x \ \underline{\text{except}} \ x = 3, -3}$

OR

$\boxed{(-\infty, -3) \cup (-3, 3) \cup (3, \infty)}$

(b) • denominator must not be 0

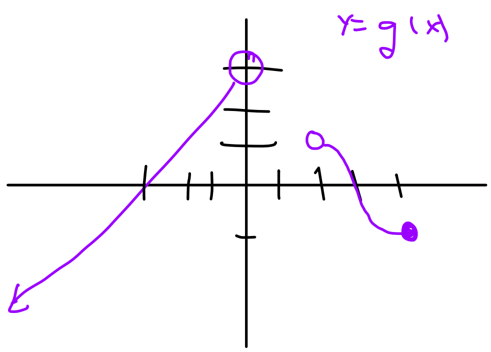
no problem — denominator is always positive

• number in square root must not be negative

need $3 - t \geq 0 \implies 3 \geq t$

Domain: $\boxed{\text{all } t \leq 3}$ OR $\boxed{(-\infty, 3]}$

<u>Ex</u> Consider $y = g(x)$ defined by



(a) Find $g(-3)$, $g(0)$, $g(4)$

(b) Find the domain and range of $g(x)$.

(a) $g(-3) = 0$, $g(0)$ undefined, $g(4) = -1$

(b) Domain: $(-\infty, 0) \cup (2, 3]$

Range: $(-\infty, 3)$

Linear Equations & Linear Functions

Def A linear equation is one that can be written in the form

no squares, cubes, roots...

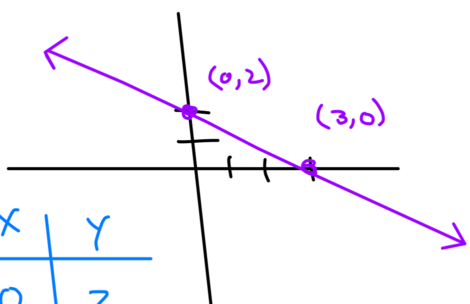
$$Ax + By = C \leftarrow \text{Standard form}$$

for some numbers A, B, C (with A, B not both zero)

* The graph of a linear equation is always a line.

Ex

(a) $2x + 3y = 6$



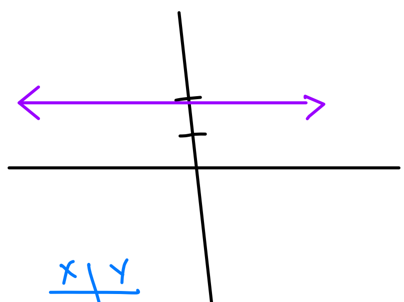
x	y
0	2
3	0

Slope: $\frac{y_2 - y_1}{x_2 - x_1} = \frac{-2}{3}$

y-intercept: $y = 2$

x-intercept: $x = 3$

(b) $11y = 22$



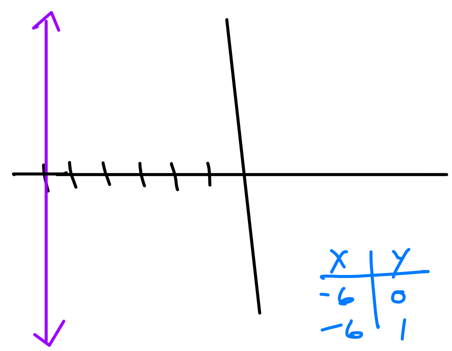
x	y
0	2
1	2

slope: 0

y-int: $y = 2$

x-int: none

(c) $-\frac{1}{2}x = 3$



x	y
-6	0
-6	1

slope: $\boxed{\text{undefined}}$

y-int: none

x-int: $x = -6$

Other forms for lines

Suppose you know the slope of a line and one point on it.

slope: m

point: (x_1, y_1)

Can you write an equation for the line?

If (x, y) is any point on the line, then

$$\frac{y - y_1}{x - x_1} = m \implies y - y_1 = m(x - x_1)$$

← point-slope form.

Now, suppose you know the slope and y-intercept.

slope: m

y-intercept: $y = b$ ← represents the point $(0, b)$

If (x, y) is any point on the line, then

$$\frac{y - b}{x - 0} = m \implies y - b = mx$$

$$\implies y = mx + b$$

← slope-intercept form

Point-slope form (section 1.5)

If a line has slope m and (x_1, y_1) is any point on the line, then an equation for the line is $y - y_1 = m(x - x_1)$

Slope-Intercept Form

If a line has slope m and y-intercept $y=b$, then an eq. for the line is

$$y = mx + b$$

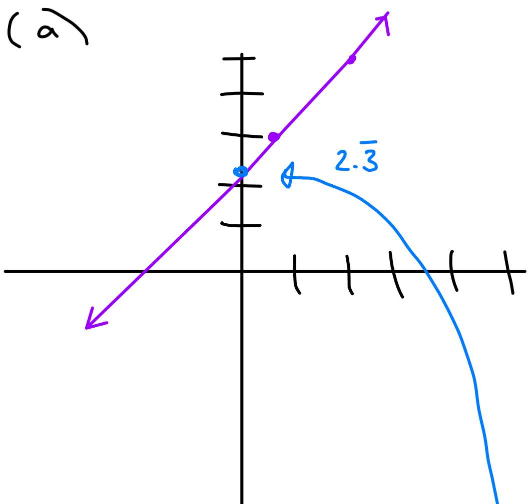
Ex Let L be the line passing through the points $(\frac{1}{2}, 3)$, $(2, 5)$.

(a) Graph L

(b) Find the slope of L

(c) Write an eq. for L in slope-intercept form.

(a)



(b)

$$m = \frac{5-3}{2-\frac{1}{2}} = \frac{2}{3/2} = \boxed{\frac{4}{3}}$$

(c) Point-slope:

$$y - 5 = \frac{4}{3}(x-2)$$

Now convert to slope-intercept

$$y = \frac{4}{3}x - \frac{8}{3} + 5$$

$$\boxed{y = \frac{4}{3}x + \frac{7}{3}}$$

Def

- <u>constant functions</u> are those of the form $f(x) = b$ for some number b
- <u>linear functions</u> are those of the form $f(x) = mx + b$ for some number $m \neq 0$

* the graphs of constant and linear functions are lines.

Average Rate of Change

If (x_1, y_1) and (x_2, y_2) are any 2 points on a line, then $\frac{y_2 - y_1}{x_2 - x_1}$ is always the same number (the slope). It describes how the line is changing.

Def Let (x_1, y_1) and (x_2, y_2) be points on the graph of any function f. The average rate of change of f on the interval $[x_1, x_2]$ is

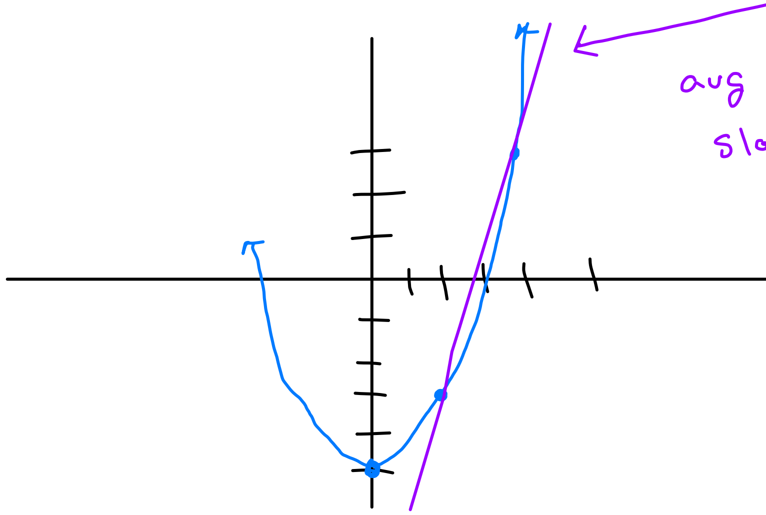
$(x_1, y_1) = (x_1, f(x_1))$, $(x_2, y_2) = (x_2, f(x_2))$

$$\text{avg. rate of change on } [x_1, x_2] = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

<u>Ex</u> Find the avg. rate of change of $g(x) = \frac{1}{2}x^2 - 5$ on $[2,4]$.

$$\frac{g(4)-g(2)}{4-2} = \frac{3-(-3)}{2} = \boxed{3}$$

* what does this represent?



avg rate of change is slope of this line

* if we modeled $g(x)$ by a <u>linear function</u> on $[2,4]$, this is the slope.

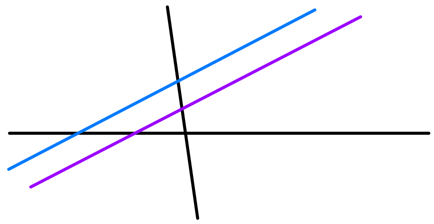
Applications of Linear Equations

$\boxed{1.5}$

Parallel and Perpendicular lines

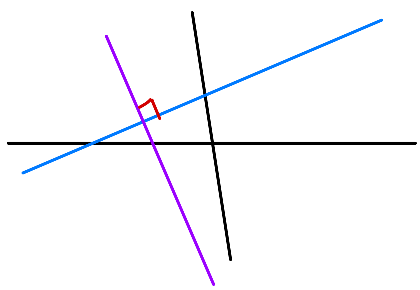
Note:

- lines L_1 and L_2 are parallel if the have the same slope: $\boxed{m_1 = m_2}$



- L_1 and L_2 are perpendicular if the meet in a right angle: $\boxed{m_1 = -\frac{1}{m_2}}$

negative reciprocal



Ex Let L be the line described by

$$x - 2y = 4$$

(a) Find an equation of a line through $(2,-3)$ and parallel to L.

(b) Find an equation of a line through $(2,-3)$ and perpendicular to L.

(a) To write an equation we need a point and the slope.

point: $(2,-3)$

slope: same a slope of L $= \frac{1}{2}$

- $x-2y=4 \iff -2y=-x+4$

  $\iff y=\frac{1}{2}x-2$ ← slope-intercept form

- slope of L is $\frac{1}{2}$

<u>Answer</u>

the line through $(2,-3)$ and parallel to L is

$$y+3=\frac{1}{2}(x-2)$$

OR

$$\boxed{y=\frac{1}{2}x-4}$$

(b) For a line through L and perp. to L ...

point: $(2,-3)$

slope: neg. reciprocal of slope of L $= -2$

<u>Answer</u>

$$y+3=-2(x-2)$$

OR

$$\boxed{y=-2x-1}$$

Linear Models

Ex suppose that during a period of drought the average water level in a pond is model by a linear function of the time since the drought began. If after 1 day of drought, the level is 6.55 ft and after 4 days of drought the level is 5.8 ft. Determine the average water level after 20 days of drought.

* let $w(x)$ be the function that outputs the average water level after x days

$w(1) = 6.55$

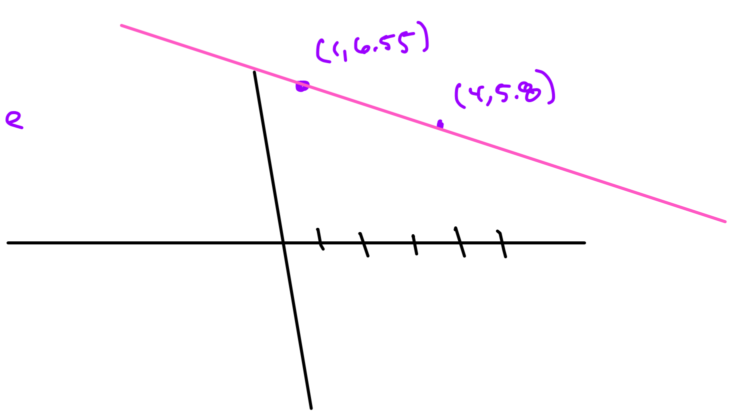
$w(4) = 5.8$

$\boxed{\text{want to find } w(20)}$

* $w(x)$ is linear, so

$y = w(x) = mx + b$ for some numbers m and b.

Let's visualize this



* write an equation for $w(x) = y$

slope: $m = \dfrac{5.8 - 6.55}{4 - 1} = -0.25$

point: $(1, 6.55)$

point-slope: $\overset{w(x)}{y} - y_1 = m(x - x_1)$

$$y - 6.55 = -0.25(x - 1)$$

simplify: $y = -0.25(x - 1) + 6.55$

$$y = -0.25x + 0.25 + 6.55$$

$$y = -0.25x + 6.8$$

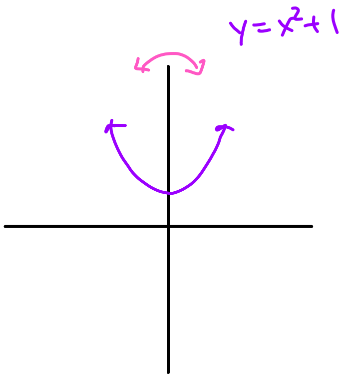
so $\underline{\underline{w(x) = -0.25x + 6.8}}$

* Use the equation to find $w(20)$

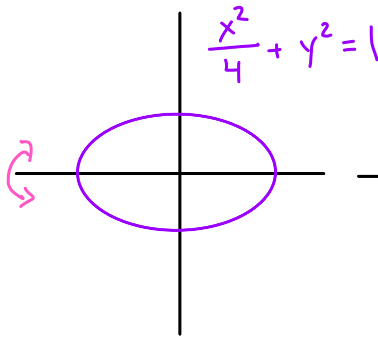
$$w(20) = -0.25(20) + 6.8 = \boxed{1.8 \text{ ft}}$$

Analyzing Graphs & More 1.7

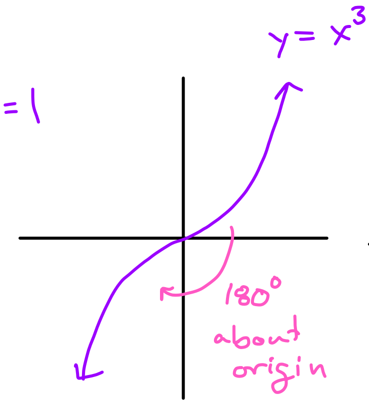
Symmetry



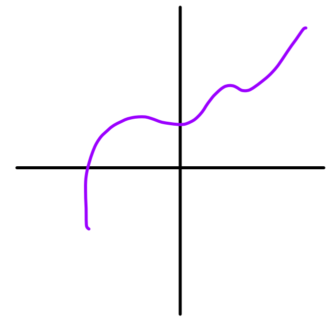
symmetric about y-axis



symmetric about x-axis



symmetric about origin



no symmetry

Algebraic test

- substituting $x \to -x$ yields an equivalent equation
- substituting $y \to -y$ yields an equivalent equation
- substituting $x \to -x$ <u>and</u> $y \to -y$ yields an equivalent equation

<u>Ex</u> Show that the graph of $y = x^3$ is symmetric about the origin but not about the y-axis.

<u>origin</u>: $x \to -x$, $y \to -y$ yields $-y = (-x)^3 \iff -y = (-1)^3 x^3$

$\iff -y = -x^3$

$\iff y = x^3$

same as original, so yes

<u>y-axis</u>: $x \to -x$ yields $y = (-x)^3 \iff y = -x^3$

is not equivalent to original (point $(1,1)$ is on original, but not on this one)

$x \to -x$ yields original

<u>Def</u>

- A function is <u>even</u> if $f(-x) = f(x)$ for all x in domain of f. (symmetric about y-axis)

OR $-f(-x) = f(x)$ so $x \to -x$, $y \to -y$ yields original

- A function is <u>odd</u> if $f(-x) = -f(x)$ for all x in domain of f. (symmetric about origin)

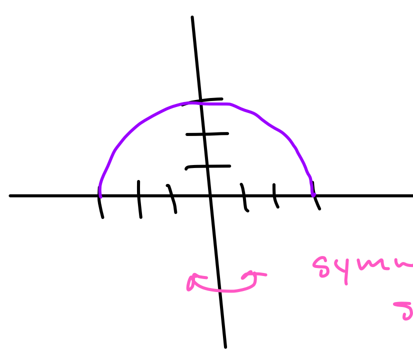
<u>Ex</u> Show that $f(x) = \sqrt{9 - x^2}$ is even. Verify by graphing.

$$\underbrace{f(-x)}_{\text{equal}} = \sqrt{9 - (-x)^2} = \sqrt{9 - x^2} = \underbrace{f(x)}_{\text{equal}}$$

To graph $y = \sqrt{9 - x^2}$, note that $y^2 = 9 - x^2$

so $x^2 + y^2 = 9$.

y must be positive, so only top of the circle



symmetric about y-axis so even

Piecewise Defined Functions

It's often the case the real world phenomena are model by different functions at different times. This leads to piecewise defined functions.

Ex Let $f(x)$ be the function

$$f(x) = \begin{cases} \boxed{-2x+1} & \text{for } x \le 0 \\ \boxed{3-x^2} & \text{for } 0 < x < 2 \\ \boxed{3x-5} & \text{for } x \geq 2 \end{cases}$$

Find each of $f(-1), f(0), f(1), f(2), f(3)$.

$f(-1) = \boxed{-2(-1)+1} = 3$

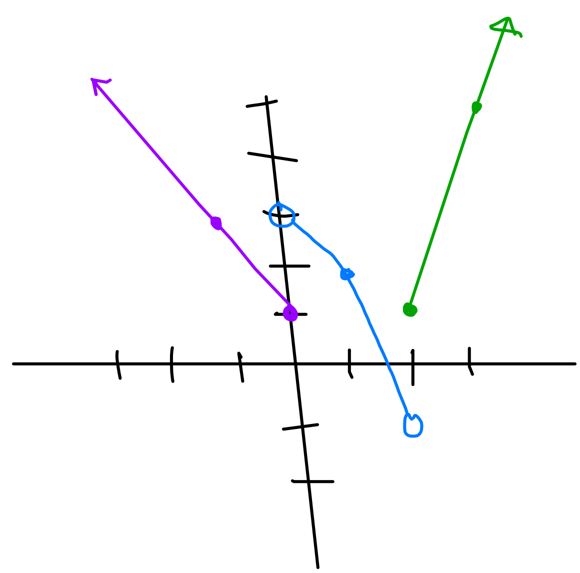
$f(0) = \boxed{-2(0)+1} = 1$

$f(1) = \boxed{3-1^2} = 2$

$f(2) = \boxed{3(2)-5} = 1$

$f(3) = \boxed{3(3)-5} = 4$

Graphically

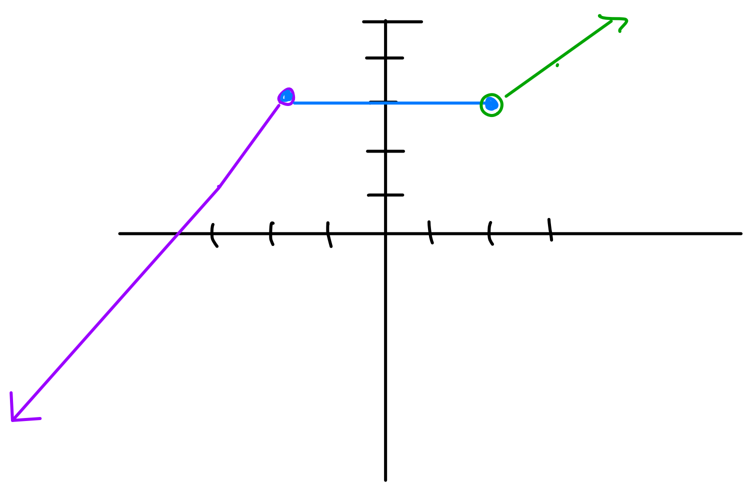


Not "continuous" b/c of jumps at $x=0$ and $x=2$.

Informal Def A function is continuous on an interval if its graph has no holes, breaks, or jumps.

Ex Graph $g(x)$ and determine if it is continuous.

$$g(x) = \begin{cases} 2x+7, & x < -2 \\ 3, & -2 \le x < 2 \\ x+1, & x > 2 \end{cases}$$



Yes, it's continuous.

Increasing / Decreasing / Maxima / Minima

$\boxed{\text{Worksheet 01}}$ see next page

01 – Increasing/Decreasing & Relative Maxima/Minima

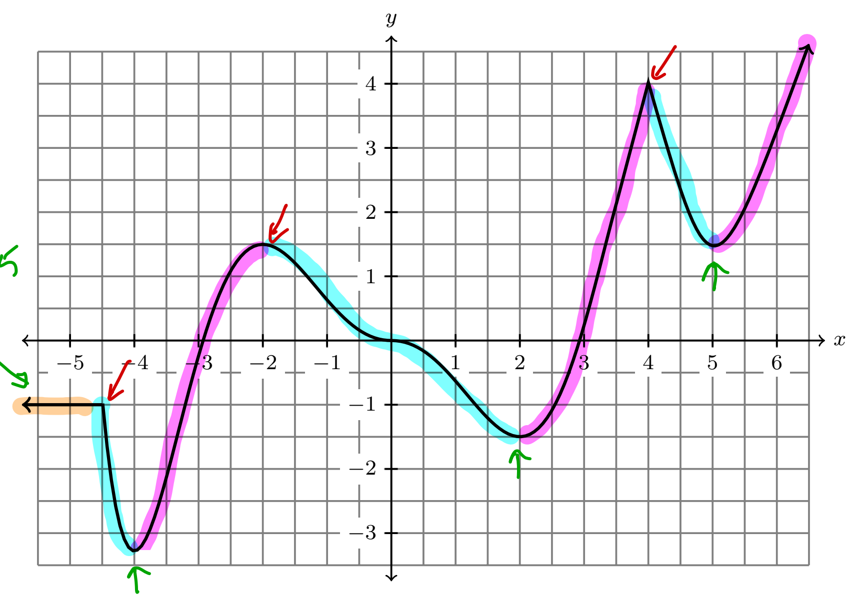
Definition: Increasing/Decreasing/Constant

Let f be a function and I an interval.

- f is **increasing** on I if $f(x_1) < f(x_2)$ for all $x_1 < x_2$. *(y-values increase from left to right.)*
- f is **decreasing** on I if if $f(x_1) > f(x_2)$ for all $x_1 < x_2$. *(y-values decrease from left to right.)*
- f is **constant** on I if if $f(x_1) = f(x_2)$ for all x_1 and x_2. *(y-values stay the same.)*

1. The graph of $f(x)$ is below.

-1 is also a rel. min looking at x=-5



(a) On what intervals is f increasing?

$(-4,-2), (2,4), (5,\infty)$

(b) On what intervals is f decreasing?

$(-4.5,-4), (-2,2), (4,5)$

(c) On what intervals is f constant?

$(-\infty,-4.5)$

Definition: Relative (or Local) Minima and Maxima

1. $f(c)$ is called a **relative minimum value** of f if $f(c) \leq f(x)$ for all x *near* c.
2. $f(c)$ is called a **relative maximum value** of f if $f(c) \geq f(x)$ for all x *near* c.

2. Let $f(x)$ be the same as in the previous problem.

(a) Find all relative minimum values of f.

-3.25 (when x = -4)
-1.5 (when x = 2)
1.5 (when x = 5)

(b) Find all relative maximum values of f.

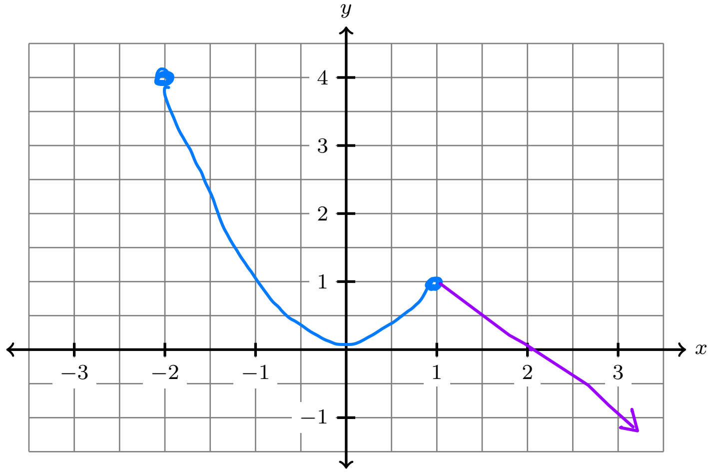
-1 (when x = -4.5)
1.5 (when x = -2)
4 (when x = 4)

3. Sketch the graph of f, and find all relative maxima and minima on its domain.

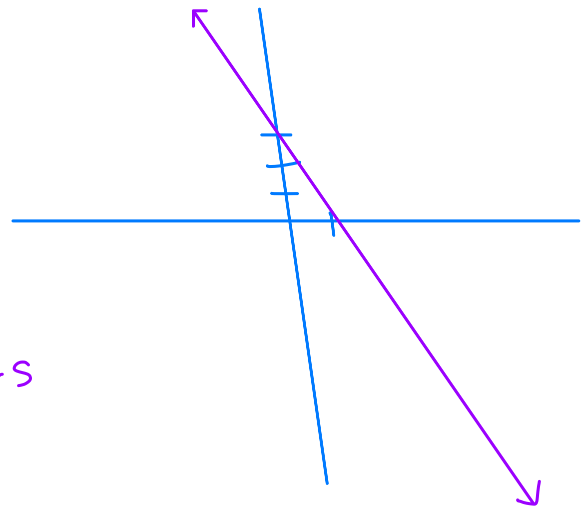
$$f(x) = \begin{cases} x^2 & \text{for } -2 \le x \le 1 \\ -x+2 & \text{for } x > 1 \end{cases}$$

rel. min at 0 (at x=0)

rel. max of 1 (at x=1)



4. Explain why $g(x) = 3 - 2x$ has no relative maxima and no relative minima.

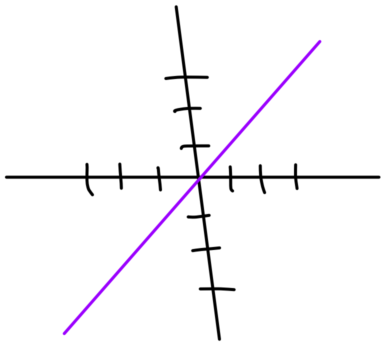


g(x) is always decreasing

Transformations of Graphs $\boxed{1.6}$

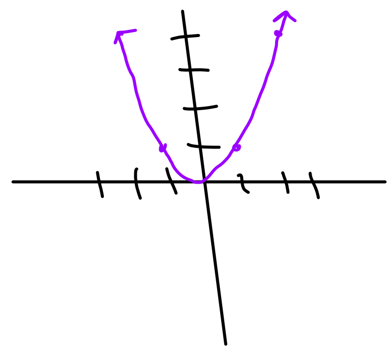
Basic Functions

$f(x) = x$



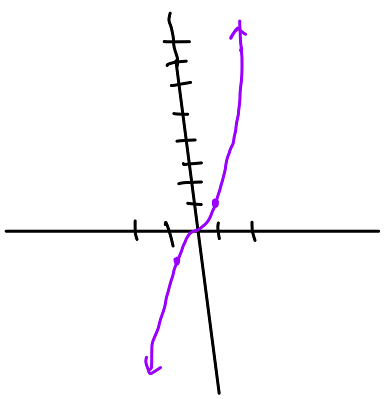
$f(x) = x^2$

x	$f(x)$ $= y$
-2	4
-1	1
0	0
1	1
2	4



$f(x) = x^3$

x	$f(x)$
-2	-8
-1	-1
0	0
1	1
2	8



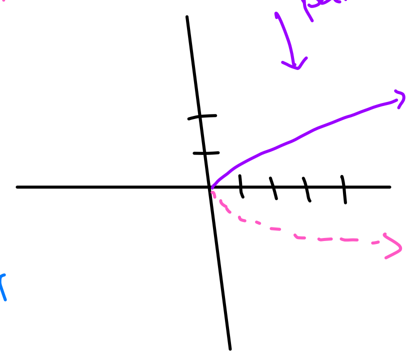
$y = \sqrt{x}$

$y^2 = x$

$f(x) = \sqrt{x}$

x	$f(x)$
-1	DNE
0	0
1	1
2	$\sqrt{2} \approx 1.4$
4	2

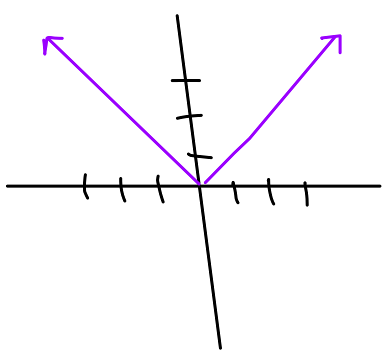
half of sideways parabola



OPTIONAL

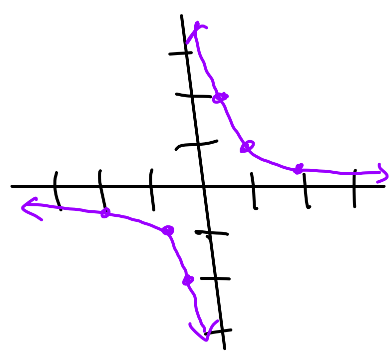
$f(x) = |x|$

x	$f(x)$
-2	2
-1	1
0	0
1	1
2	2



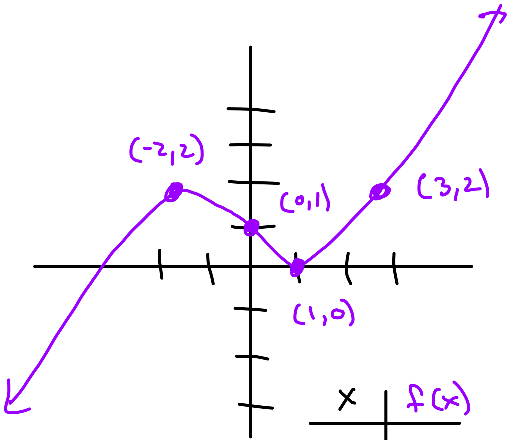
$f(x) = \frac{1}{x}$

x	$f(x)$
-2	$-\frac{1}{2}$
-1	-1
$-\frac{1}{2}$	-2
0	0
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$



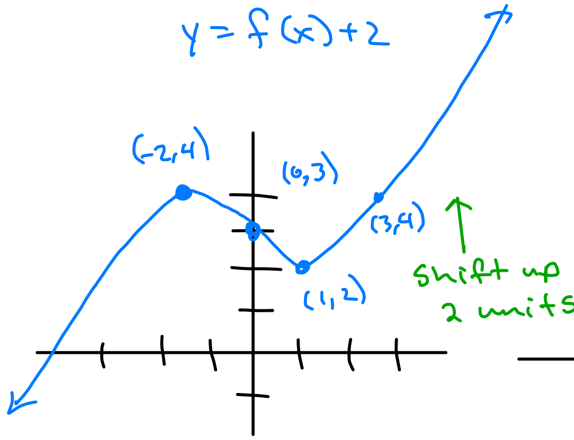
Vertical & Horizontal Translations (Shifts)

$y=f(x)$



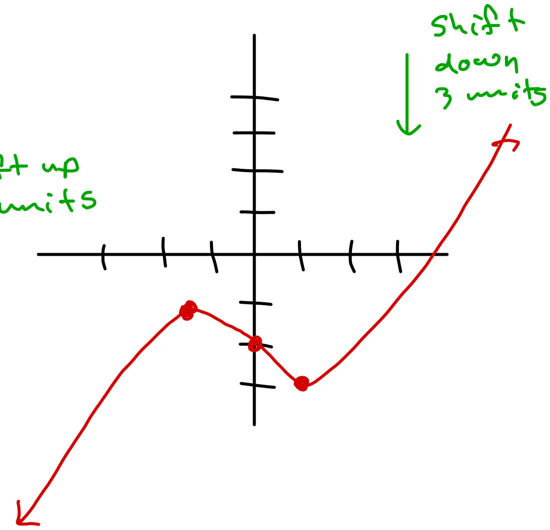
x	$f(x)$
-2	2
0	1
1	0
3	2

$y=f(x)+2$



x	$f(x)+2$
-2	$2+2$
0	$1+2$
1	$0+2$
3	$2+2$

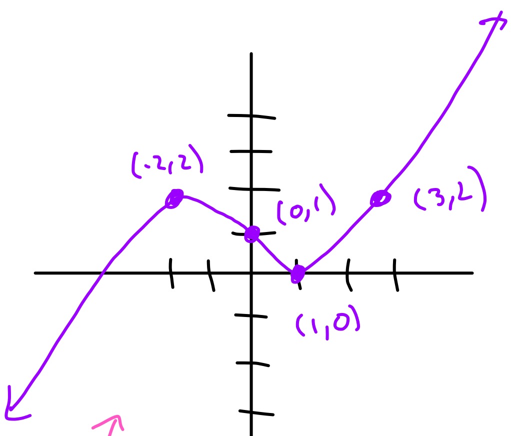
$y=f(x)-3$



Thm The graph of $y=f(x)+k$ is the graph of $y=f(x)$ shift up | down by k units

if $k>0$ if $k<0$

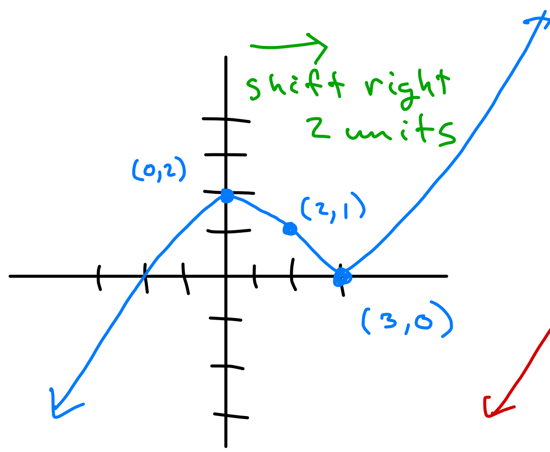
$y=f(x)$



To Josh
use $y=x^2$
next time

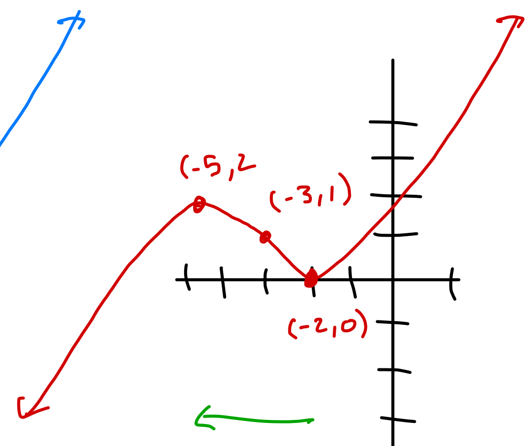
x	$f(x)$
-2	2
0	1
1	0
3	2

$y=f(x-2)$



x	$x-2$	$f(x-2)$
0	-2	2
2	0	1
3	1	0
5	3	2

$y=f(x+3)$



shift
left 2 units

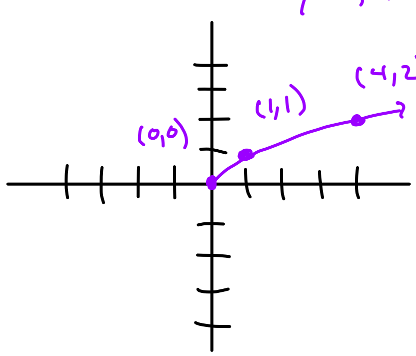
Thm The graph of $y = f(x+h)$ is the graph of $y = f(x)$ shifted

- right if h is negative
- left if h is positive

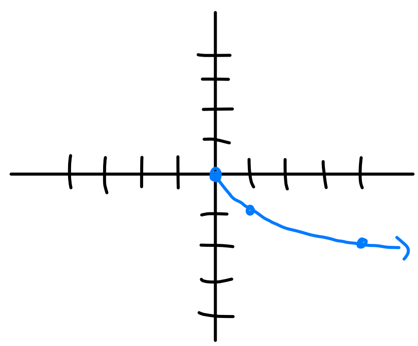
! You can always plot a couple of points to check.

Reflections

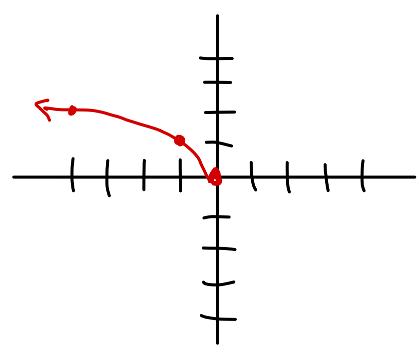
$y = f(x)$



$y = -f(x)$



$y = f(-x)$



x	$f(x)$
0	0
1	1
4	2

x	$-f(x)$
0	-0
1	-1
4	-2

x	$-x$	$f(-x)$
0	0	0
-1	1	1
-4	4	2

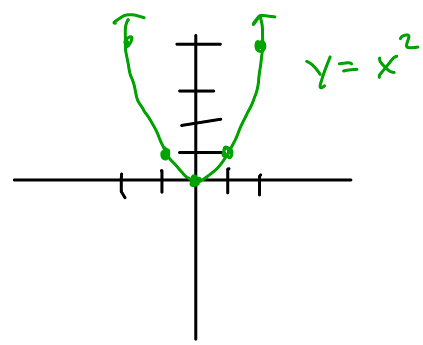
Thm

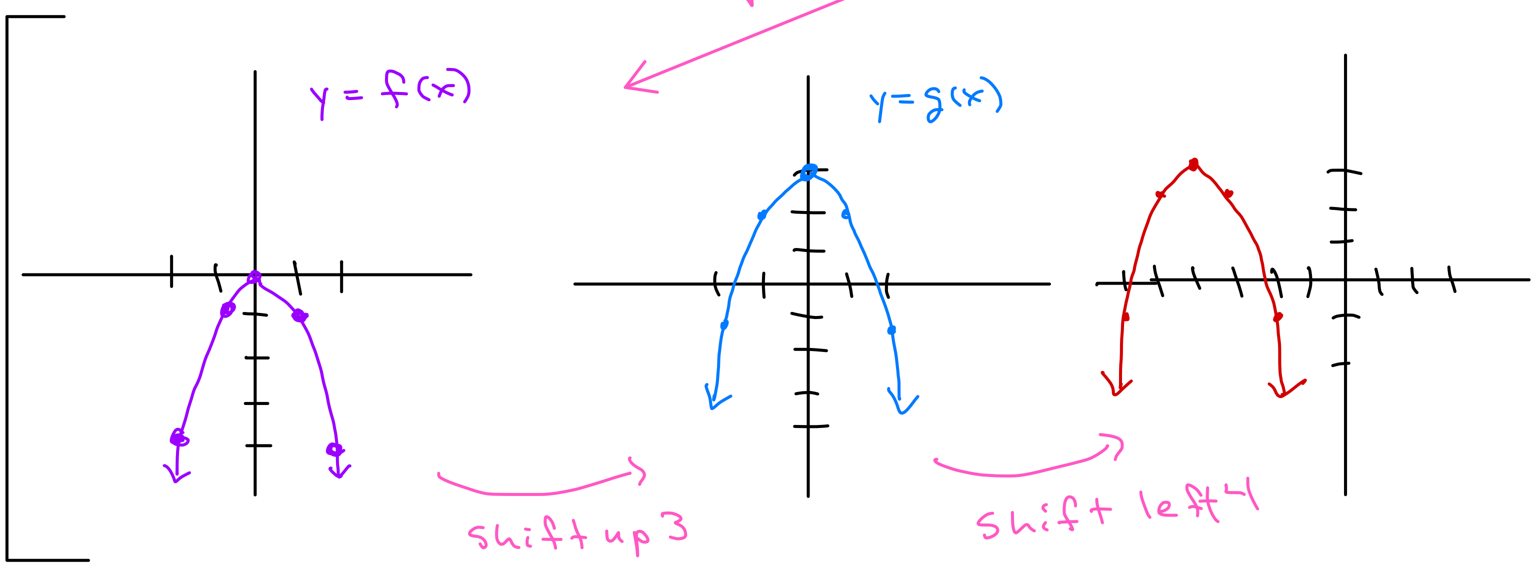
1. The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected over the x-axis
2. The graph of $y = f(-x)$ is the graph of $y = f(x)$ reflected over the y-axis.

Ex Graph each of the following.

1. $f(x) = -x^2$
2. $g(x) = 3 - x^2$
3. $h(x) = 3 - (x+4)^2$

Recall:

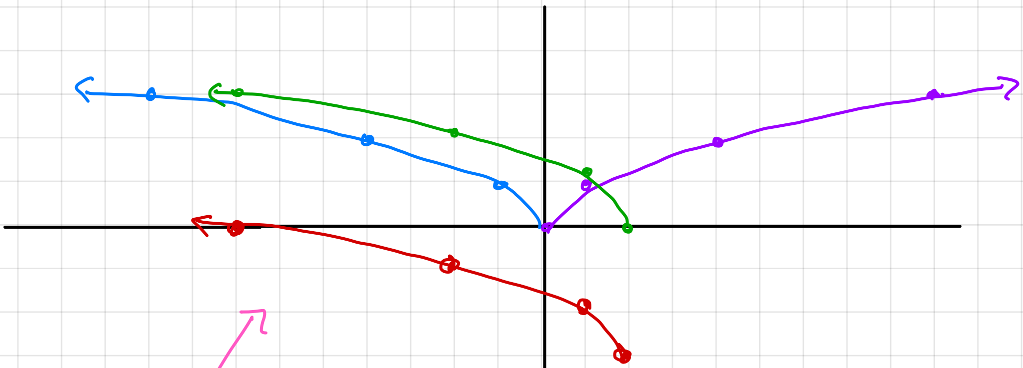




Ex Find a sequence of translations and reflections to transform $f(x) = \sqrt{x}$ into $g(x) = \sqrt{2-x} - 3$. Then graph $g(x)$.

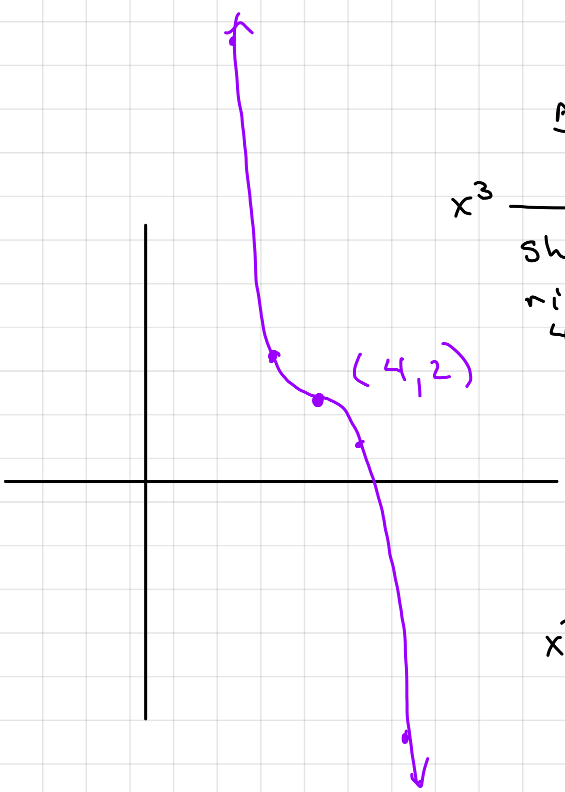
one possibility is

$$\sqrt{x} \xrightarrow[\text{over y-axis}]{\text{reflect}} \sqrt{-x} \xrightarrow[\text{by 2}]{\text{shift right}} \sqrt{-(x-2)} \xrightarrow[\text{by 3}]{\text{shift down}} \sqrt{-x+2} - 3$$



red is the final answer

Ex The function $g(x)$ below has been translated and reflected from $f(x) = x^3$. Find an equation for $g(x)$.



Attempt 1

$$x^3 \xrightarrow[\text{right 4}]{\text{shift}} (x-4)^2 \xrightarrow[\text{up 2}]{\text{shift}} (x-4)^2+2 \xrightarrow[\text{over y-axis}]{\text{reflect}} -((x-4)^2+2)$$

no doesn't work.

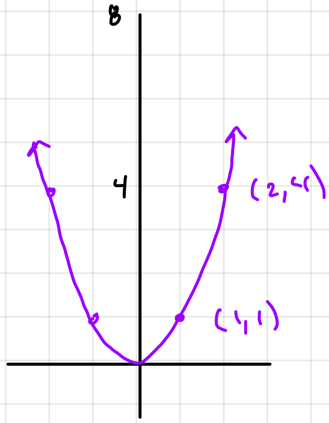
Attempt 2

$$x^3 \xrightarrow[\text{over x-axis}]{\text{reflect}} -x^3 \xrightarrow[\text{up 2}]{\text{shift}} -x^3+2 \xrightarrow[\text{right 4}]{\text{shift}} \boxed{-(x-4)^3+2}$$

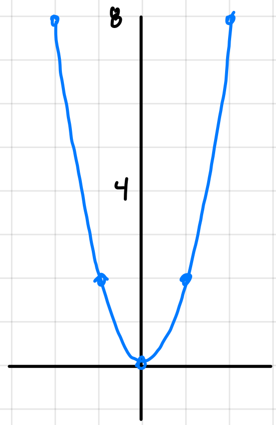
Shrinking & Stretching

Ex Graph each of the following.

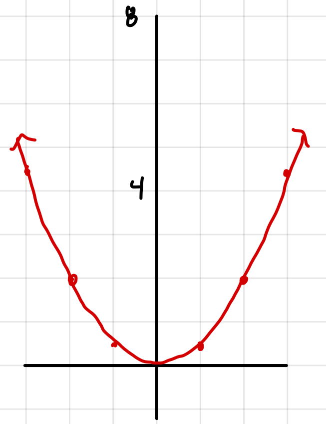
(a) $f(x) = x^2$



(b) $g(x) = 2x^2$



(c) $h(x) = \frac{1}{2}x^2$



x	$f(x)$
-2	4
-1	1
0	0
1	1
2	4

x	$2 \cdot f(x)$
-2	$2 \cdot 4 = 8$
-1	$2 \cdot 1 = 2$
0	$2 \cdot 0 = 0$
1	$2 \cdot 1 = 2$
2	$2 \cdot 4 = 8$

x	$\frac{1}{2} \cdot f(x)$
-2	$\frac{1}{2} \cdot 4 = 2$
-1	$\frac{1}{2} \cdot 1 = \frac{1}{2}$
0	$\frac{1}{2} \cdot 0 = 0$
1	$\frac{1}{2} \cdot 1 = \frac{1}{2}$
2	$\frac{1}{2} \cdot 4 = 2$

Thm Let a be a positive number. The graph of $y = af(x)$ is the graph of $y = f(x)$ stretched vertically by a factor of a.

$0 < a < 1$ results in a shrinking.

Similarly,

Thm Let a be positive. The graph of $y = f(ax)$ is the graph of $y = f(x)$ stretched horizontally by a factor of $\frac{1}{a}$. ← reciprocal!

When stretching vertically or horizontally it's best to plot points to avoid mistakes

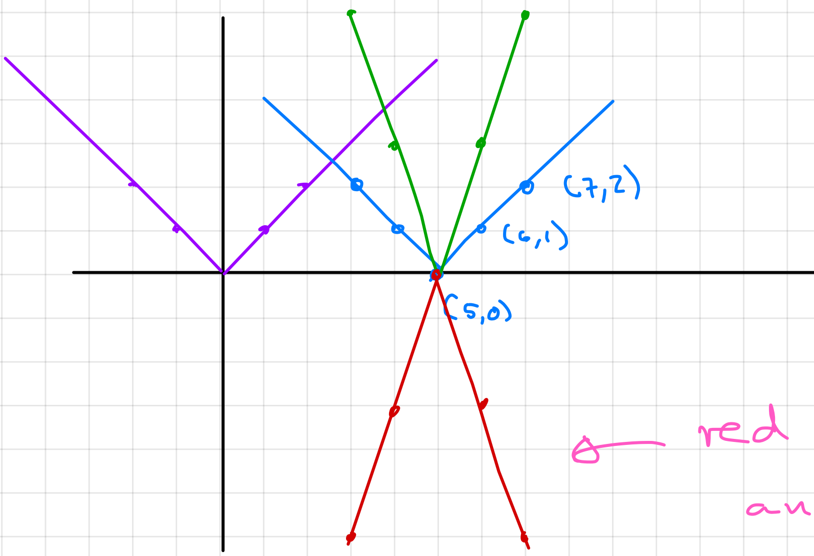
Ex Graph $h(x) = -3|x-5|$

$$|x| \xrightarrow{} |x-5| \xrightarrow{} 3|x-5| \xrightarrow{} -3|x-5|$$

shift right 5

stretch vertically by 3

reflect over x-axis



x	$3\|x-5\|$
3	6
4	3
5	0
6	3
7	6

← red is final answer

Ex Where is the vertex of the parabola $f(x) = 7-(x+5)^2$? Does the parabola open up or down?

Algebra of functions (A.K.A. Combining functions) $\boxed{1.8}$

$(f+g)(x)$ $(f-g)(x)$ $(f\cdot g)(x)$ $\left(\frac{f}{g}\right)(x)$

We can add, subtract, multiply, and divide functions to get new functions. We can also compose...

Composition

$(f\circ g)(x)$ means $f(g(x))$, or in a picture

$$x \longrightarrow \boxed{g} \longrightarrow g(x) \longrightarrow \boxed{f} \longrightarrow f(g(x))$$

* The domain of $(f\circ g)$ is all x in the domain of g for which $g(x)$ is also in the domain of f.

Ex Use the tables to compute $(f\circ g)(4)$ and $(g\circ f)(5)$

x	$f(x)$
-1	2
0	3
5	0

x	$g(x)$
-1	3
0	7
4	-1

$(f\circ g)(4) = f(g(4)) = f(-1) = \boxed{2}$

$(g\circ f)(5) = g(f(5)) = g(0) = \boxed{7}$

Ex Let $f(x) = x^2 - 9x$, $g(x) = \sqrt{2x}$, $h(x) = \frac{1}{5+x}$

(a) Find $(f+g)(8) = f(8) + g(8) = (64-72) + \sqrt{16} = \boxed{-4}$

(b) Find $\left(\frac{f}{g}\right)(4) = \frac{f(4)}{g(4)} = \frac{16-36}{2\sqrt{2}} = \frac{-10}{2\sqrt{2}} = \boxed{\frac{-5}{\sqrt{2}}}$

(c) Find $f(h(-2)) = f(h(-2)) = f\left(\frac{1}{3}\right) = \frac{1}{9} - 3 = \boxed{\frac{-26}{9}}$

Ex with f, g, h as before

(a) Find $\left(\frac{g}{f}\right)(x)$ and determine the domain

(b) Find $(f\circ h)(x)$ " " " "

(c) Find $(h\circ h)(x)$ " " " "

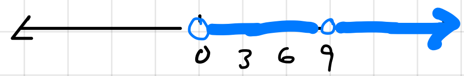
$$f(x) = x^2 - 9x, \quad g(x) = \sqrt{2x}, \quad h(x) = \frac{1}{5+x}$$

(a) $\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \boxed{\frac{\sqrt{2x}}{x^2 - 9x}}$

Domain

① $2x \geq 0 \Rightarrow x \geq 0$

② $\left.\begin{array}{l} x^2 - 9x \neq 0 \\ x^2 - 9x = x(x-9) \end{array}\right\} \Rightarrow x \neq 0, 9$



$\boxed{\text{Domain: } (0, 9) \cup (9, \infty)}$

(b) $(f \circ h)(x) = f(h(x)) = \boxed{\left(\frac{1}{5+x}\right)^2 - \frac{9}{5+x}}$

OR

$$\frac{1}{(5+x)^2} - \frac{9 \cdot (5+x)}{(5+x)^2} = \frac{-44 - 9x}{(5+x)^2}$$

Domain $\boxed{x \neq -5}$ OR $\boxed{(-\infty, -5) \cup (-5, \infty)}$

(c) $(h \circ h)(x) = h(h(x)) = \frac{1}{5 + \frac{1}{5+x}} = \frac{1}{\frac{5(5+x)}{5+x} + \frac{1}{5+x}}$

$$= \frac{1}{\frac{26 + 5x}{5+x}} = \boxed{\frac{5+x}{26+5x}}$$

Domain

① $26 + 5x \neq 0 \Rightarrow x \neq \frac{-26}{5}$

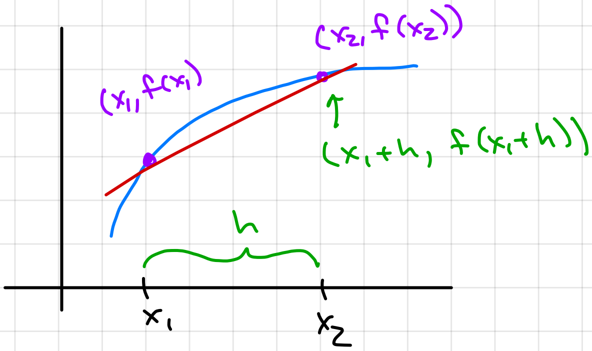
not in domain of h

② $5 + x \neq 0 \Rightarrow x \neq -5$

$\boxed{\text{Domain: } x \neq -5, \frac{26}{5}}$

Difference Quotients

these measure the same quantity



$x_2 = x_1 + h$

$$\text{Average change over } [x_1, x_2] = \frac{f(x_2)-f(x_1)}{x_2 - x_1}$$

OR

$$= \frac{f(x_1+h)-f(x_1)}{x_1+h-x_1}$$

$$= \frac{f(x_1+h)-f(x_1)}{x_1}$$

$\frac{f(x+h)-f(x)}{h}$ is called the difference quotient

Ex Find the difference quotient of each of the following, simplifying as much as possible.

(a) $f(x) = -4x^2 - 2x + 6$

optional → (b) $g(x) = \frac{1}{x}$

(a)

$$\frac{f(x+h)-f(x)}{h} = \frac{-4(x+h)^2 - 2(x+h) + 6 - (-4x^2 - 2x + 6)}{h}$$

$$= \frac{-4(x^2+2xh+h^2) - \cancel{2x} - 2h + \cancel{6} + 4x^2 + \cancel{2x} - \cancel{6}}{h}$$

$$= \frac{\cancel{-4x^2} - 8xh - 4h^2 - 2h \cancel{+4x^2}}{h}$$

$$= \frac{-8xh - 4h^2 - 2h}{h}$$

$$= \frac{\cancel{h}(-8x-4h-2)}{\cancel{h}} = \boxed{-8x-2-4h}$$

(b)

$$\frac{g(x+h)-g(x)}{h} = \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \frac{\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{h}$$

$$= \frac{\frac{x-x-h}{x(x+h)}}{h}$$

$$= \frac{\frac{-h}{x(x+h)}}{\frac{h}{1}}$$

$$= \frac{-\cancel{h}}{x(x+h)} \cdot \frac{1}{\cancel{h}} = \boxed{\frac{-1}{x(x+h)}}$$