03 – Rational Functions & End Behavior

Definition: Limiting behavior at infinity

1. $x \to \infty$ means that "x increases without bound"
   - $x \to \infty$ means we are considering x-values like $100, 1000, 10000, \ldots$
2. $x \to -\infty$ means that "x decreases without bound"
   - $x \to -\infty$ means we are considering x-values like $-100, -1000, -10000, \ldots$

1. Let $f(x) = \dfrac{3x+8}{2x}$. Answer the following by plugging in several appropriate x-values.

(a) As $x \to \infty$, $f(x) \to$ _____

(b) As $x \to -\infty$, $f(x) \to$ _____

2. Let $g(x) = \dfrac{7}{x}$. Answer the following by plugging in several appropriate x-values.

(a) As $x \to \infty$, $g(x) \to$ _____

(b) As $x \to -\infty$, $g(x) \to$ _____

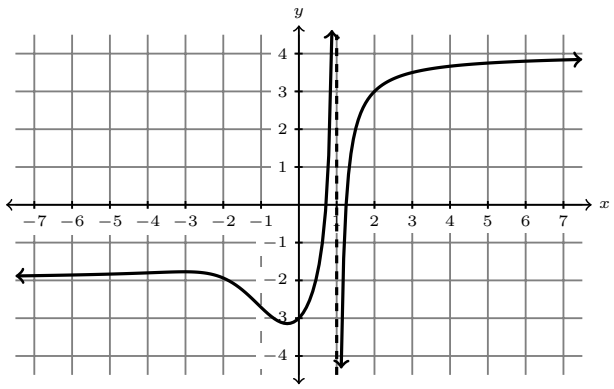
Definition: Horizontal Asymptote

A line $y = L$ is a **horizontal asymptote** of $y = f(x)$ if *at least one* of the following are true:

- As $x \to \infty$, $f(x) \to L$
- As $x \to -\infty$, $f(x) \to L$

3. What are the vertical asymptotes of the functions in the first two problems?

4. Find all vertical and horizontal asymptotes of the graph given below.



(a) Vert. asymptotes: _____

(b) Hor. asymptotes: _____

Strategy: Finding Horizontal Asymptotes of Rational functions

Suppose that

$$f(x) = \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}.$$

Then, as $x \to \infty$ or $x \to -\infty$, $x \to \infty$, $f(x) \approx \dfrac{a_n x^n}{b_m x^m}$, so

1. if $\deg(\text{TOP}) > \deg(\text{BOTTOM})$, then as $x \to \infty$ and $x \to -\infty$, $f(x) \to \pm\infty$

- f has no horizontal asymptote;

2. if $\deg(\text{TOP}) < \deg(\text{BOTTOM})$, then as $x \to \infty$ and $x \to -\infty$, $f(x) \to 0$

- $y = 0$ is the horizontal asymptote of f;

3. if $\deg(\text{TOP}) = \deg(\text{BOTTOM})$, then as $x \to \infty$ and $x \to -\infty$, $f(x) \to \dfrac{a_n}{b_m}$;

- $y = \dfrac{a_n}{b_m}$ is the horizontal asymptote of f.

5. Find all horizontal and vertical asymptotes of each of the following rational functions.

(a) $f(x) = \dfrac{x+3}{2x^2-7}$

(b) $g(x) = \dfrac{2x^2+8x+8}{7+5x^2}$

(c) $h(x) = \dfrac{x^3-1}{x^2+4x-5}$