2.1 Tangent + Velocity Problems

GW-01

Q: But, what do we mean by the (instantaneous) velocity of an object? Velocity is a rate of change, but at a particular instant, nothing is changing!?

Instantaneous velocity will be defined to be the "limiting value" of the average velocities, i.e.

$$v(1) = \lim_{t \to 1} \frac{h(t) - h(1)}{t - 1}$$

More later!!

* So, limits are unavoidable!

A Geometric Connection

GW-02

The slope of the tangent line (hence the line itself) will be defined as the "limiting value" of the slopes of the secant lines.

* limits are unavoidable!

2.2 Limit of a function

$\boxed{\text{GW-03}}$ # 1, 2

$\boxed{\text{GW-03}}$ Limit def, # 3

have them identify confusing parts

$\boxed{\text{GW-03}}$ One-sided def, # 4, 5, 6

Notice that...

<u>Thm</u> $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ <u>AND</u> $\lim_{x \to a^+} f(x) = L$.

$\boxed{\text{GW-04}}$ Inf. limit def, # 1

$\boxed{\text{GW-04}}$ V.A def, # 2 – 5

2.3 Limit Laws

Q: what is $\lim_{x \to 2} \frac{7x^3}{5-x}$?

... well as $x \to 2$, $5-x \to 3$ and $x^3 \to 8$, so $\frac{7x^3}{5-x} \to \frac{56}{3}$

... but how do we justify this?

Limit Laws Suppose that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$.

1. and 2. $\lim_{x \to a} \left[\underset{L}{f(x)} \pm \underset{M}{g(x)} \right] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = L \pm M$

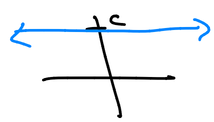
3. $\lim_{x \to a} \left[c \underset{L}{f(x)} \right] = c \lim_{x \to a} f(x) = cL$, for any constant

4. $\lim_{x \to a} \left[f(x) \cdot g(x) \right] = L \cdot M$

5. $\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = \frac{L}{M}$ if $M \neq 0$.

6. $\lim_{x \to a} (f(x))^n = L^n$

7. $\lim_{x \to a} c = c$



(others)

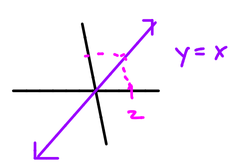
11. $\lim_{x \to a} \sqrt[n]{\underset{L}{f(x)}} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L}$

provided $L > 0$ when n is even.

Ex Use the limit laws to find $\lim_{x \to 2} \frac{7x^3}{5-x}$.

$$\underset{5}{=} \frac{\lim_{x \to 2} 7x^3}{\lim_{x \to 2} (5-x)} \underset{2,3}{=} \frac{7 \lim_{x \to 2} (x^3)}{\lim_{x \to 2} 5 - \lim_{x \to 2} x} \underset{6,7}{=} \frac{7 \left(\lim_{x \to 2} x \right)^3}{5 - \lim_{x \to 2} x} = \frac{7 \cdot 2^3}{5-2} = \frac{56}{3}$$

use laws to break into building blocks



Nice... but don't really want to that again...

polynomial / polynomial

Direct Substitution If f is a rational function and a is in the domain of f, then $\lim_{x\to a} f(x) = f(a)$.

just plug in

* so, DS tells us that $\lim_{x\to -1} \frac{x^2-1}{x-1} = \frac{0}{-2} = 0$
* But, DS does Not apply to $\lim_{x\to -1} \frac{x^2-1}{x+1}$ ... see why?

Ex Find $\lim_{x\to -1} \frac{x^2-1}{x+1}$.

* DS does Not apply — it yields $\frac{0}{0}$

Not an answer!

Try to manipulate function.

$$\lim_{x\to -1} \frac{x^2-1}{x+1} = \lim_{x\to -1} \frac{(x-1)(x+1)}{x+1} = \lim_{x\to -1} (x-1) = \boxed{-2}$$

b/c limit ignores $x=-1$

! $\frac{(x-1)(x+1)}{x+1} = x-1$ whenever $x \neq -1$

GW-05 Strategy, # 1 (a), (b), (c)

Ex Find $\lim_{h\to 0} \frac{\frac{1}{1+h} - 1}{h}$.

* DS yields $\frac{0}{0}$, so try to manipulate.

$$\lim_{h\to 0} \frac{\frac{1}{1+h} - 1}{h} = \lim_{h\to 0} \frac{\frac{1}{1+h} - 1}{h} \cdot \frac{(1+h)}{(1+h)}$$

clear denominator

... or combine fractions in numerator

$$= \lim_{h\to 0} \frac{1-(1+h)}{h(1+h)}$$

limit ignores 0

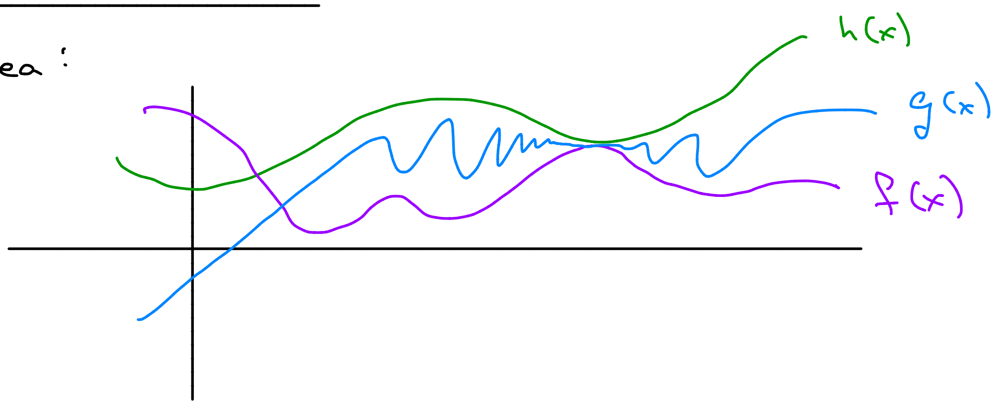
$$= \lim_{h\to 0} \frac{-h}{h(1+h)}$$

$$= \lim_{h\to 0} \frac{-1}{1+h} = \boxed{-1}$$

$\boxed{\text{GW-04}}$ #1 (d)(e)(f)

Squeeze Theorem

Idea:



Theorem If

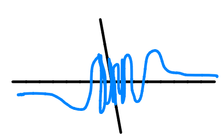
- $f(x) \le g(x) \le h(x)$ for all x near a (except possibly at a)

AND

- $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L$

then $\lim_{x\to a} g(x) = L$.

Ex Recall $\lim_{x\to 0^+} \sin\left(\frac{1}{x}\right)$ DNE



But what about $x\sin\left(\frac{1}{x}\right)$?

Find $\lim_{x\to 0^+} x\sin\left(\frac{1}{x}\right)$.

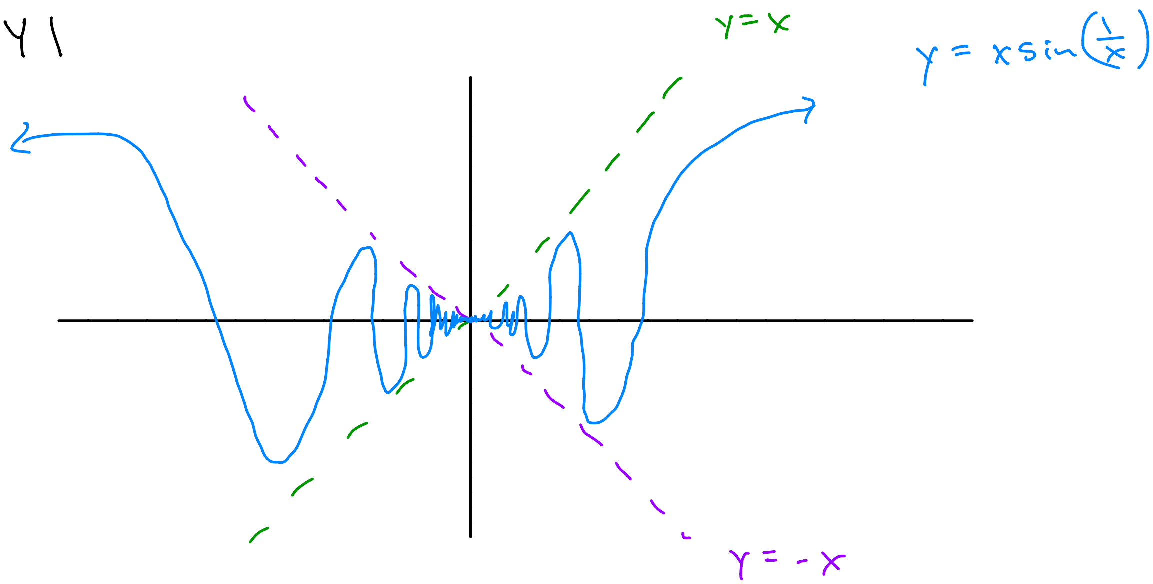
* what to do? DS yields $0 \cdot \sin\left(\frac{1}{0}\right)$?? simplify??

Try Squeeze Thm. Note that

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1 \implies \overset{0}{-x} \le x\sin\left(\frac{1}{x}\right) \le \overset{0}{x} \quad (x>0)$$

Since $\lim\limits_{x\to0^+} -x = \lim\limits_{x\to0^+} x = 0$, the Squeeze Thm tells us that $\lim\limits_{x\to0^+} x\sin\left(\frac{1}{x}\right) = 0$.

FYI



2.5 Continuity

Q: What do you think it means for a function to be continuous (at a point)?

$\boxed{\text{GW-06}}$ #1 quick look

Continuity (Intuitive def.) A function is continuous if you can draw its graph without lifting your pen. That is, the graph has NO

- $\ast\ \lim_{x\to a} f(x) \neq f(a)$ • holes ⟵ removable discontinuities
- $\ast\ \lim_{x\to a} f(x)$ not a real # • breaks ⟵ infinite discontinuities (V.A.s)
- $\ast\ \lim_{x\to a} f(x)$ DNE • jumps ⟵ jump discontinuities
- • other weird stuff

$\boxed{\text{GW-06}}$ #1, 2 together

$\boxed{\text{GW-06}}$ #3

$\boxed{\text{GW-06}}$ #4, 5 together

Theorem (Continuity Laws) If f, g are continuous at a, then so are $f+g$, $f-g$, $f\cdot g$, and $\frac{f}{g}$ provided $g(a) \neq 0$. If g is continuous at a and f is continuous at $g(a)$, then $f\circ g$ is continuous at a too.

$\ast$ The final point requires the following limit law...

Theorem If $\lim_{x\to a} g(x) = b$ and if f is <u>continuous</u> at b, then $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(b)$.

Theorem (continuous Building Blocks) The following are continuous on their domains:

1. Rational functions (and polynomials)
2. Root functions
3. trig. and inverse trig. functions
4. exponential and logarithmic functions

Ex Determine where $\ln(1-\sqrt{x})$ is continuous.

* $\ln(1-\sqrt{x})$ is built up from functions that are continuous on their domains, so $\ln(1-\sqrt{x})$ is continuous on its domain.

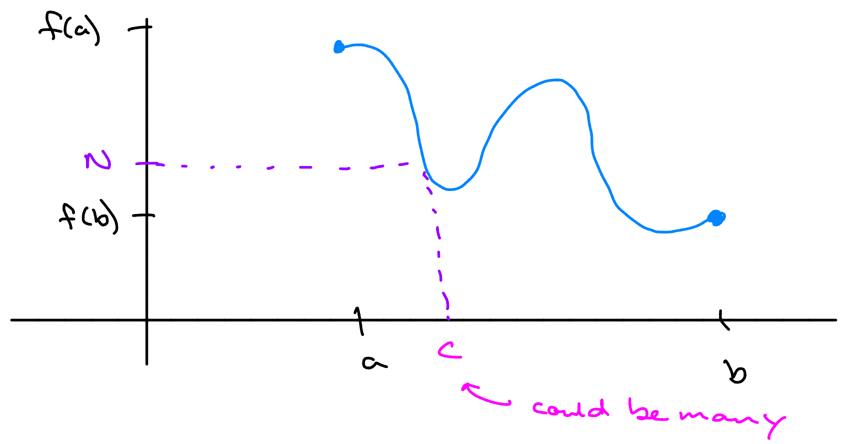
* So what is the domain of $\ln(1-\sqrt{x})$?

Need: $1-\sqrt{x} > 0$ and $\sqrt{x} \geq 0$

$\Rightarrow x \geq 0$ and $x < 1$ so $\boxed{0 \leq x < 1}$

Intermediate Value Theorem

IVT Suppose f is <u>continuous</u> on $[a,b]$ and that N is any number b/w $f(a)$ & $f(b)$. Then there is at least one value c b/w a and b s.t. $f(c) = N$.



Ex Does $x^4 - x - 1 = 0$ have any real solutions?

* can you factor? graph? ... try IVT

* Let $f(x) = x^4 - x - 1$ ... want a c s.t. $f(c) = 0$ ← $\underline{\underline{N}}$

IVT applies!!
(i) f is continuous everywhere (b/c it is a poly.)
(ii) $f(0) = -1 < 0$ $\quad f(2) = 13 > 0$

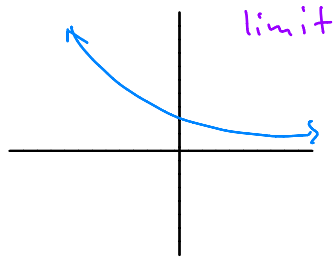
Thus, by IVT, there is a c b/w 0 and 2 s.t.
$f(c) = 0$.

2.6 Limits at Infinity

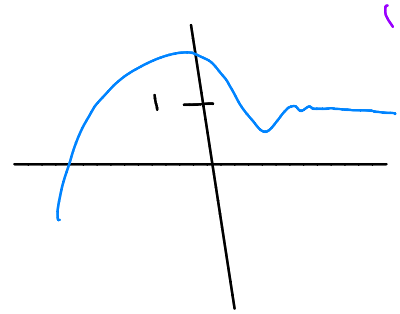
Q: What should $\lim_{x \to \infty} f(x) = L$ mean?

Q: Given the graphs below, how would you answer

$$\lim_{x \to \infty} f(x) = \underline{\ ?\ }$$



limit = 0



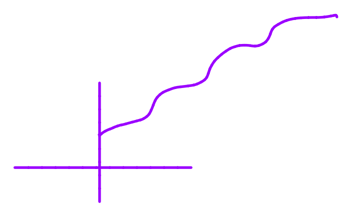
limit = 1

$\boxed{\text{GW-07}}$ # 1

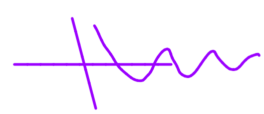
$\boxed{\text{GW-07}}$ # 2

<u>Ex</u> Find the following

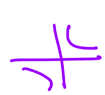
(a) $\lim_{x \to \infty} (\overset{\infty}{x} + \overset{-1 \le \ \le 1}{\cos x}) = \infty$



(b) $\lim_{x \to \infty} \left(\overset{0}{\frac{1}{x}} + \cos x\right) = \text{DNE}$



(c) $\lim_{x \to 0^+} \cos\left(\frac{1}{\ln(1 + \frac{1}{x})}\right) = \cos(0) = 1$



$\frac{1}{x}$

Algebraic techniques

Ex Find the following

Indeterminate!!

(a) $\lim_{x\to\infty} \dfrac{2x^3+x+1}{7+5x^4} \cdot \dfrac{1/x^4}{1/x^4} = \lim_{x\to\infty} \dfrac{\cancel{\frac{2}{x}}^{\to 0} + \cancel{\frac{1}{x^3}}^{\to 0} + \cancel{\frac{1}{x^4}}^{\to 0}}{\cancel{\frac{7}{x^4}}^{\to 0} + 5} = \boxed{0}$

$\frac{\infty}{-\infty}$

highest power in denom.

(b) $\lim_{x\to\infty} \dfrac{2x^3+x+1}{\sqrt{7+5x^4}} \cdot \dfrac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \dfrac{2x+\frac{1}{x}+\frac{1}{x^2}}{\sqrt{7+5x^4}\cdot\frac{1}{\sqrt{x^4}}}$

$= \lim_{x\to\infty} \dfrac{\cancel{2x}^{\to\infty} + \cancel{\frac{1}{x}}^{\to 0} + \cancel{\frac{1}{x^2}}^{\to 0}}{\sqrt{\cancel{\frac{7}{x^4}}^{\to 0} + 5}}$

$= \text{"}\dfrac{\infty}{\sqrt{5}}\text{"} = \boxed{\infty}$

same idea:

denom $\approx \sqrt{x^4} = x^2$

highest power

$\boxed{\text{GW-07}}$ #3

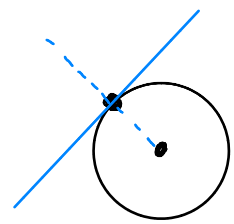
Q: what did you notice in these examples? might you be able to anticipate some of the answers?

– please provide evidence of your suggestion.

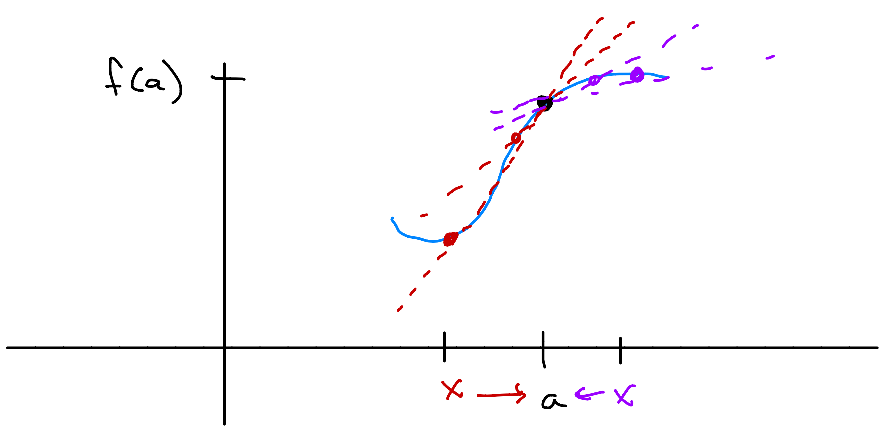
2.7 Derivatives & Rates of Change

Tangent lines

The line tangent to a circle at a point P is easy to define...



we now try to define the line tangent to an arbitrary curve $y = f(x)$ at a point $P = (a, f(a))$. we start by looking at secant lines...

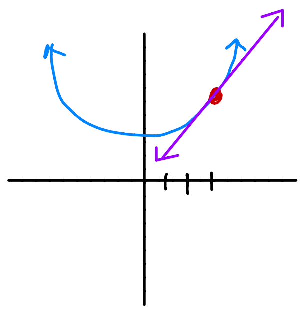


$\boxed{\text{GW-08}}$ #1

$\boxed{\text{GW-08}}$ Def. of tan. line, #2

Ex Draw the tan. line to $y = x^2 + 1$ at the point where $x = 3$, and find an eq. for the tan. line.

Graph



Tan. line

- slope:

$$m = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$$
$$= \lim_{h \to 0} \frac{(3+h)^2 + 1 - (3^2+1)}{h}$$
$$= \lim_{h \to 0} \frac{9 + 6h + h^2 - 9}{h}$$
$$= \lim_{h \to 0} (6+h) = 6$$

- point: $(3, f(3)) = (3, 10)$

Answer:

$$y - 10 = 6(x-3)$$

OR

$$y = 6(x-3) + 10$$

$\boxed{\text{GW-08}}$ #3

Instantaneous Velocity

Think back to the beginning of the course...

$\boxed{\text{GW-08}}$ Def. of velocity, #4

The derivative of a function

This is the unifying idea of finding slopes of tan. lines, finding velocities, ...

$\boxed{\text{GW-09}}$ Def of derivative

Summary

① $f'(a)$ is the slope of the tan. line to $y=f(x)$ at $x=a$

② If $f(x)$ is position at time x, then $f'(a)$ is velocity at time a.

position $\xrightarrow{\text{der.}}$ velocity $\xrightarrow{\text{der}}$ acceleration

... in general

Def If $y=f(x)$, then the (instantaneous) <u>rate of change of y with respect to x</u> at a is $f'(a)$.

* Why? $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ ← limit of average rates of change

* Note: units for $f'(a) = \frac{\text{units for } f(x)}{\text{units for } x}$

$\boxed{\text{GW-09}}$ #1

2.8 The Derivative Function

Recall: GW-09 Def. of der. function

Ex Let $f(x) = \frac{1}{x}$.

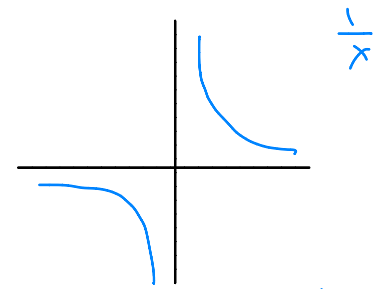
① Find a formula for $f'(x)$.

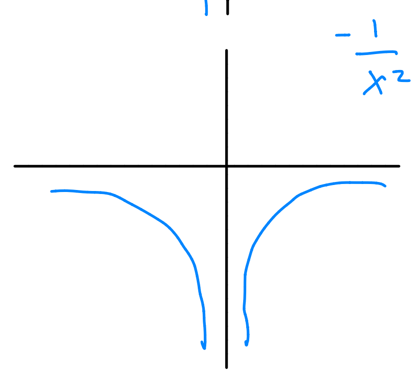
② use your formula to find $f'(-2)$ and $f'(1)$.

③ Graph $f(x)$ and $f'(x)$.

①
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \quad \frac{(x+h)x}{(x+h)x}$$
$$= \lim_{h\to 0} \frac{x - (x+h)}{h(x+h)\cdot x}$$
$$= \lim_{h\to 0} \frac{-h}{h(x+h)x}$$
$$= \lim_{h\to 0} \frac{-1}{(x+\cancel{h}^{\,0})x} = \boxed{\frac{-1}{x^2}}$$

③





②
$$f'(-2) = \frac{-1}{(-2)^2} = -\frac{1}{4}$$
$$f(1) = \frac{-1}{1^2} = -1$$

Q: Does it make sense that $f'(x) < 0$?

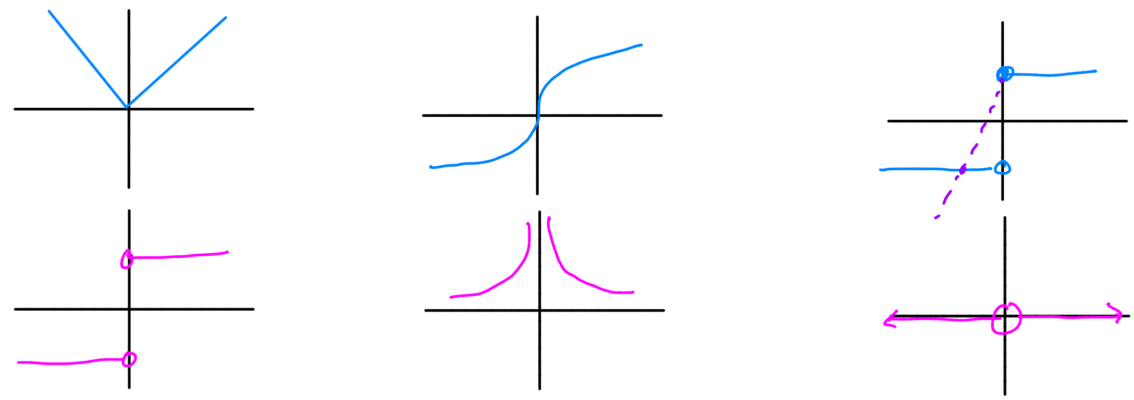
GW-09 #2

GW-09 Def. differentiability, #3

Non-differentiability

Ex Sketch f and f'

(a) $f(x) = |x|$ (b) $f(x) = \sqrt[3]{x}$ (c) $f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$

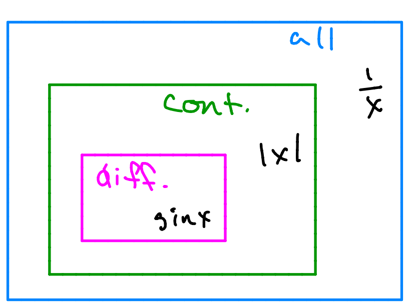


❗ Graphically, f is <u>NOT</u> differentiable at a if

1. it has a (sharp) corner at $x = a$
2. it has a vertical tangent line at $x = a$
3. it has a discontinuity at $x = a$

Thm If f is differentiable at a, then f is continuous at a.

Picture



pf Assume $f'(a)$ exists. We want to show $\lim_{x \to a} f(x) = f(a)$ <u>or</u> equivalently that $\lim_{x \to a} (f(x) - f(a)) = 0$. Observe,

$$\lim_{x \to a} (f(x) - f(a)) = \lim_{x \to a} \underbrace{\frac{f(x) - f(a)}{x - a}}_{f'(a)} \cdot \underbrace{(x - a)}_{0}$$

why?

$$= f'(a) \cdot 0 = 0$$

<u>Notation</u> Let $y = f(x)$.

① $\frac{d}{dx}$ means "take the derivative with respect to x"

- $\frac{d}{dx}(f(x)) = f'(x)$
- $\frac{d}{dx}(y)$ is usually written $\frac{dy}{dx}$
- Thus, $f'(x) = \frac{dy}{dx}$.
- $f'(a) = \left.\frac{dy}{dx}\right|_{x=a}$

② Second derivative

- f'' means $(f')'$
- $\frac{d^2y}{dx^2}$ means $\frac{d}{dx}\left(\frac{dy}{dx}\right)$
- Thus, $f''(x) = \frac{d^2y}{dx^2}$