

3.1 Derivatives of polys. and exp. functions

Idea: We want to speed up our computation of derivatives by proving formulas

Q: What is the der. of $f(x) = -3$?

Graphically



Slope is always $\underline{\underline{0}}$

Algebraically

$$\begin{aligned}f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\&= \lim_{h \rightarrow 0} \frac{-3 - (-3)}{h} = \boxed{0}\end{aligned}$$

So, $f(x) = C$ implies $f'(x) = 0$.

Q: What about der. of x, x^2, x^3, \dots also x^{-1}, x^{-2}, \dots

from before

$$\left\{ \begin{array}{l} \bullet \frac{d}{dx}(x) = 1 \\ \bullet \frac{d}{dx}(x^2) = 2x \\ \bullet \frac{d}{dx}(x^{-1}) = -x^{-2} \end{array} \right. \Rightarrow \frac{d}{dx}(x^n) = ?$$

GW-10

Constant + Power Rules, # 1

So, we know $(x^n)' = nx^{n-1}$, but what about
?

GW-10

Sum-Diff - const. mult. Rules

Ex Find the der. of $f(x) = 5x^3 - x^2 + \sqrt{x} + \frac{7}{x} - 11$

- First convert to powers

$$f(x) = 5x^3 - x^2 + x^{1/2} + 7x^{-1} - 11$$

- Then

$$f'(x) = \boxed{15x^2 - 2x + \frac{1}{2}x^{-1/2} - 7x^{-2} + 0}$$

GW-10

#2, #3

Exponential Functions

We know how to find $\frac{d}{dx}(x^2)$. What about $2^x \dots$ or e^x ?

Let $f(x) = e^x$. Let's find $f'(x)$.

- First, let's try to find $f'(0)$.

GW-10 #4

So, we think $f'(0) = 1$. This means

$$1 = f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{e^h - 1}{h}$$

In fact,

Def e is defined to be the number s.t. $\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$.

(2) Now,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \rightarrow 0} e^x \cdot \underbrace{\left(\frac{e^h - 1}{h} \right)}_{\text{purple}} = e^x. \end{aligned}$$

GW-10 #5

3.2 Product & Quotient Rules

You know

$$\circ (x^3)' = 3x^2$$

$$\circ (e^x)' = e^x$$

So, what about $(x^3 e^x)'$?

Experiment:

$$\begin{array}{ccc}
 (x^3 \cdot x^4)' & & \\
 \swarrow ?? & & \searrow \\
 (x^3)' \cdot (x^4)' & & (x^7)' \\
 \downarrow & & \downarrow \\
 3x^2 \cdot 4x^3 & & 7x^6 \\
 \downarrow & & \downarrow \\
 12x^5 & \xrightarrow{\text{not equal}} &
 \end{array}$$

GW-11 Product Rule

Ex Compute

$$(a) (x^3 e^x)' = \textcircled{1} \textcircled{2}' + \textcircled{2} \textcircled{1}' = x^3 (e^x)' + e^x (x^3)' = \boxed{x^3 e^x + e^x \cdot 3x^2}$$

$$\begin{aligned}
 (b) \left[\left(x - \frac{1}{\sqrt[5]{x}} \right) (e^x - 3x^2) \right]' &= \left(x - x^{-\frac{1}{5}} \right)' (e^x - 3x^2) + (e^x - 3x^2)' \left(x - x^{-\frac{1}{5}} \right) \\
 &= \boxed{\left(x - x^{-\frac{1}{5}} \right)' (e^x - 6x) + (e^x - 3x^2) \left(1 + \frac{1}{5} x^{-\frac{6}{5}} \right)}
 \end{aligned}$$

$\frac{1}{x^{\frac{1}{5}}} = x^{-\frac{1}{5}}$

GW-11 #1, 2

↙ prod. rule

So, you know how to find $\left[(1-x+\sqrt{x})(3e^x) \right]'$.

What about

$$\left[\frac{3e^x}{1-x+\sqrt{x}} \right]' ?$$

GW-11 Quotient Rule ⚡ the jingle.

Ex Compute

$$(a) \left[\frac{3e^x}{1-x+\sqrt{x}} \right]' = \frac{(1-x+\sqrt{x})(3e^x)' - 3e^x(1-x+\sqrt{x})'}{(1-x+\sqrt{x})^2}$$
$$= (1-x+\sqrt{x})(3e^x) - 3e^x(-1 + \frac{1}{2}x^{-\frac{1}{2}})$$

$$(b) \left[\frac{x^3 e^x}{x^2+7} \right]' = \frac{(x^2+7)(x^3 e^x)' - x^3 e^x(x^2+7)'}{(x^2+7)^2}$$
$$= \frac{(x^2+7)(3x^2 e^x + e^x \cdot x^3) - x^3 e^x \cdot 2x}{(x^2+7)^2}$$

GW-11 #3, 4

Proof of Product Rule

$$\begin{aligned}
 (f(x) \cdot g(x))' &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{f(x+h)g(x+h)} - \cancel{f(x+h)g(x)} + \cancel{f(x+h)g(x)} - \cancel{f(x)g(x)}}{h} \\
 &= \lim_{h \rightarrow 0} f(x+h) \left(\frac{\cancel{g(x+h)} - \cancel{g(x)}}{h} \right) + g(x) \left(\frac{\cancel{f(x+h)} - \cancel{f(x)}}{h} \right) \\
 &= f(x)g'(x) + g(x)f'(x)
 \end{aligned}$$

□

Proof of Quotient Rule

Let $h(x) = \frac{f(x)}{g(x)}$. Then $h(x) \cdot g(x) = f(x)$.

$$(PR) \quad h(x)g'(x) + g(x)h'(x) = f'(x)$$

$$\implies \frac{f(x)}{g(x)} \cdot g'(x) + g(x) \left(\frac{f(x)}{g(x)} \right)' = f'(x)$$

$$\implies \left(\frac{f(x)}{g(x)} \right)' = \frac{f'(x) - \frac{f(x)}{g(x)}g'(x)}{g(x)} \cdot \frac{g(x)}{g(x)}$$

$$\implies \left(\frac{f(x)}{g(x)} \right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

3.3 Trig. Derivatives

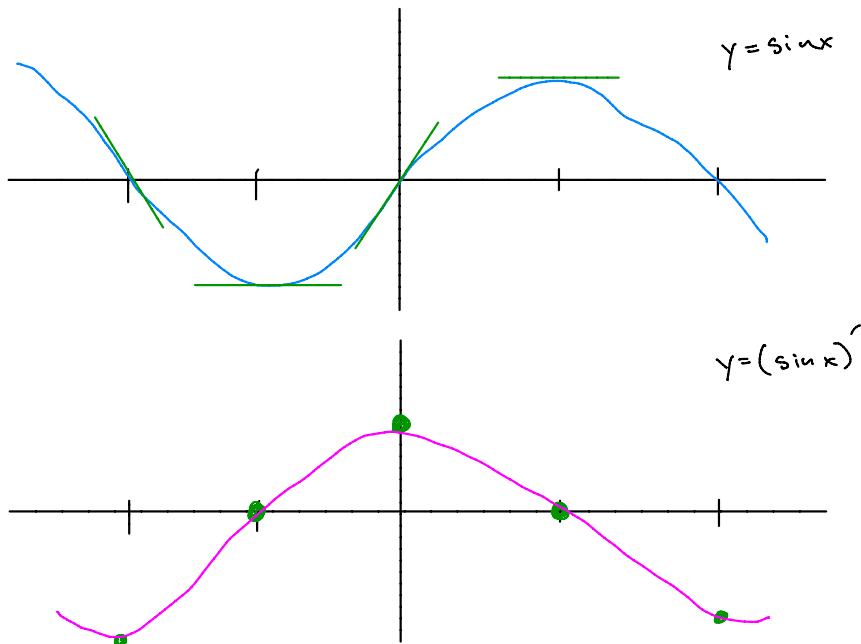
Q: what is $(\sin x)'$?

⚠ Our convention is that x is in radians.

Experiment

GW-12

#1



* $(\sin x)'$ looks like $\cos x$!

Aside: Special limits

You found $\frac{d}{dx} (\sin x) \Big|_{x=0} = 1$ so $\lim_{h \rightarrow 0} \frac{\sin(h) - \sin(0)}{h} = 1$

Similarly, $\frac{d}{dx} (\cos x) \Big|_{x=0} = 0$ so $\lim_{h \rightarrow 0} \frac{\cos(h) - \cos(0)}{h} = 0$

Thus...

Theorem

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad \text{AND} \quad \lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = 0$$

Thm $(\sin x)' = \cos x$

Pf

$$\begin{aligned} (\sin x)' &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin x \cosh h + \sinh \cos x - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin x \cosh h - \sin x}{h} + \frac{\sinh \cos x}{h} \\ &= \lim_{h \rightarrow 0} \sin x \left(\frac{\cosh h - 1}{h} \right) + \cos x \frac{\sinh}{h} \\ &= \cos x \quad \square \end{aligned}$$

Thm $(\cos x)' = -\sin x$

Ex Find $(\tan x)'$.

$$\begin{aligned} (\tan x)' &= \left(\frac{\sin x}{\cos x} \right)' \text{ OR } \frac{\cos x(-\cos x) - \sin x \sin x}{\cos^2 x} \\ &= \frac{-1}{\cos^2 x} = \boxed{\sec^2 x} \end{aligned}$$

WS-12

Der. of trig. functions

WS-12

#2

3.4 Chain Rule

Q: Can you compute $(\cos(x))'$?

Q: What about $(\cos(x^2))'$?

() this is about composition

WS-13 Chain Rule

Some comments:

$$\circ (\underline{f(g(x))})' = \underline{f'(g(x))} \cdot \underline{g'(x)}$$

der. of outside
 (and plug inside
 back in)

der. of inside

$$\circ \text{ let } \boxed{u = g(x)}. \text{ Then}$$

$$(\underline{f(g(x))})' = (\underline{f(u)})' = \underline{f'(u)} \cdot \underline{u'}$$

Ex Find the derivatives.

$$(a) f(x) = \cos \frac{x^2}{u}$$

$$f'(x) = (\cos(u))' = -\sin(u) \cdot u' = \boxed{-\sin(x^2) \cdot 2x}$$

$$(b) g(x) = \frac{(x^2 e^x)^{17}}{u}$$

$$g'(x) = (u^{17})' = 17u^{16} \cdot u' = \boxed{17(x^2 e^x)^{16} \cdot (x^2 e^x + e^x \cdot 2x)}$$

WS-13 #1, 2

Ex Find $\frac{d}{dx}(e^{17x^2})$.

$$(e^{\frac{17x^2}{u}})' = e^{u'} = e^u \cdot u' = e^{17x^2} \cdot 34x$$

Thm $\frac{d}{dx}(e^u) = e^u \cdot u'$ so $\frac{d}{dx}(e^{ax}) = e^{ax} \cdot a$.

Thm If $a > 0$, then $(a^x)' = a^x \cdot \ln a$

Pf $(a^x)' = (e^{\ln(a^x)})' = (e^{x \ln a})' = e^{x \ln a} \cdot \ln a = a^x \cdot \ln a$

Ex (harder) Find der. of $\tan^3(1+5^x)$

$$\begin{aligned} \tan^3 z &= (\tan z)^3, \\ (\tan^3(1+5^x))' &= \left(\left(\tan(1+5^x) \right)^3 \right)' \\ &= (u^3)' \\ &= 3u^2 \cdot u' \\ &= 3\tan^3(1+5^x) \cdot (\tan(1+5^x))' \\ &= 3\tan^3(1+5^x) \sec^2(1+5^x) \cdot (1+5^x)' \\ &= 3\tan^3(1+5^x) \sec^2(1+5^x) (5^x \cdot \ln 5). \end{aligned}$$

Idea of chain rule

① Let $h(x) = f(g(x))$. we must show

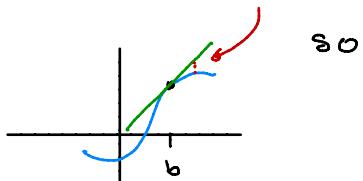
$$h'(a) = f'(g(a)) \cdot g'(a)$$

Now,

$$h'(a) = \lim_{h \rightarrow 0} \frac{f(g(a+h)) - f(g(a))}{h}$$

② Linear approximation: the tangent line to $f(x)$ at any $x=b$ is

close together $y = f'(b)(x-b) + f(b)$



$$f(x) \approx f'(b)(x-b) + f(b) \quad \text{when } x \text{ is near } b$$

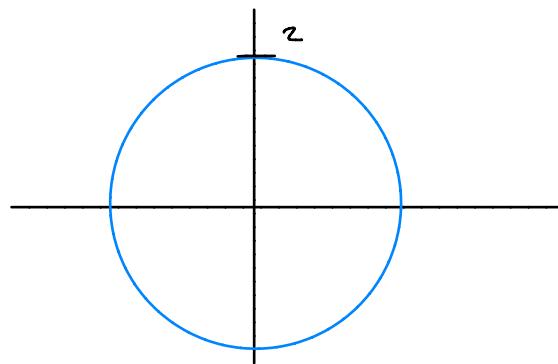
③ Since $g(a+h)$ is near $g(a)$

$$\begin{aligned} h'(a) &= \lim_{h \rightarrow 0} \frac{f(g(a+h)) - f(g(a))}{h} \\ &= \lim_{h \rightarrow 0} \frac{f'(g(a)) (g(a+h) - g(a)) + f(g(a)) - f(g(a))}{h} \\ &= \lim_{h \rightarrow 0} f'(g(a)) \cdot \frac{g(a+h) - g(a)}{h} \\ &= f'(g(a)) \cdot g'(a) \quad \square \end{aligned}$$

3.5 Implicit Differentiation

Implicitly Defined Functions

The eq. $y = x^2$ or $y = \sqrt{x} - 7$ define y explicitly as a function of x . The eq. $x^2 + y^2 = 4$ does not explicitly define a function. Why?



if $x=0$ is the input, there are 2 outputs:
 $y = \pm 2$

However, $x^2 + y^2 = 4$ does implicitly define y as a function of x , but it depends on what part of the circle we use:

$$\text{Top: } y \geq 0 \quad y = \sqrt{4-x^2}$$

$$\text{Bottom: } y \leq 0 \quad y = -\sqrt{4-x^2}$$

* $x^2 + y^2 = 4$ also defines x as a function of y

$$\text{Right: } x \geq 0 \quad x = \sqrt{4-y^2}$$

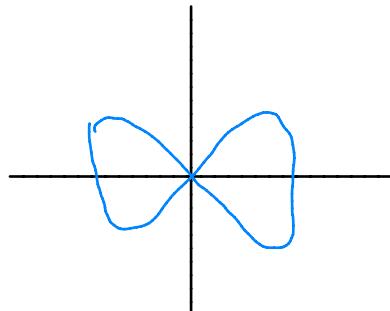
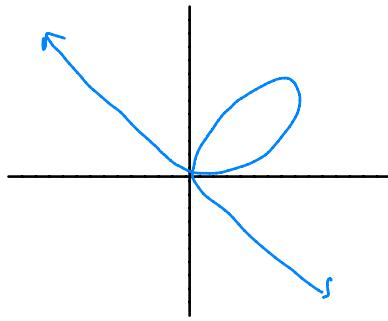
$$\text{Left: } x \leq 0 \quad x = -\sqrt{4-y^2}$$

* However, we usually focus on y as a function of x .

Similarly, consider equations like

$$x^3 + y^3 = 6 \times y$$

$$y^2 = x^2 - x^4$$



For another pretty one, consider

$$\sin(xy) = 1 + \cos(y^2)$$

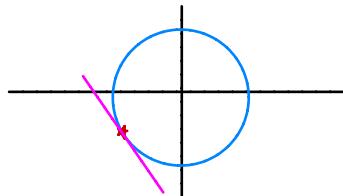
Desmos Me!

⚠ working with equations (instead of explicit functions) allows us a lot more flexibility and it often makes things a lot less messy.

Q: how to we do calculus on equations?

A: work implicitly...

Ex Suppose that $x^2 + y^2 = 4$. Find $\frac{dy}{dx}$ and use this to find the tan. line to $x^2 + y^2 = 4$ at $(-1, -\sqrt{3})$. ($\theta = 4\pi/3$)



we work implicitly.

- Think $y = f(x)$
(but we do not solve for y)

Steps to find $\frac{dy}{dx}$

① Take der. of both sides remembering $y = f(x)$.

$$x^2 + y^2 = 4$$

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(4)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(4)$$

$$2x + 2y \frac{dy}{dx} = 0$$

chain rule !!

$$\frac{d}{dx}(f(x))^2$$

$$= 2f(x) \cdot f'(x)$$

② Solve for $\frac{dy}{dx}$

$$2y \frac{dy}{dx} = -2x \implies \boxed{\frac{dy}{dx} = -\frac{x}{y}}$$



x's { y's - okay!

Now, to find the tan. line:

Point: $(-1, -\sqrt{3})$

$$\text{Slope: } \left. \frac{dy}{dx} \right|_{(-1, -\sqrt{3})} = \left. \frac{-x}{y} \right|_{(-1, -\sqrt{3})} = -\frac{1}{\sqrt{3}}$$

$$\text{the line: } \boxed{y + \sqrt{3} = -\frac{1}{\sqrt{3}}(x + 1)} \quad \text{or} \quad y = -\frac{1}{\sqrt{3}}(x+1) - \sqrt{3}$$

Ex Find $\frac{dy}{dx}$ if $\sin(xy) = 1 + \cos(y^2)$, and use it to find $\left. \frac{dy}{dx} \right|_{(2\pi, \pi)}$.

* I'll use prime notation this time

① Der. at both sides remembering $y = f(x)$

$$[\sin(xy)]' = [1 + \cos(y^2)]'$$

$$\cos(xy) \cdot (xy)' = -\sin(y^2) \cdot (y^2)'$$

$$\cos(xy)(xy' + y) = -\sin(y^2) \cdot 2y y'$$

② solve for y'

$$\begin{aligned} \cos(xy) \cdot xy' + \cos(xy) \cdot y &= -2yy' \cdot \sin(y^2) \\ \underline{\underline{\cos(xy) \cdot xy'}} + \underline{\underline{2yy' \cdot \sin(y^2)}} &\stackrel{\text{collect } y'}{=} -\cos(xy) \cdot y \end{aligned}$$

$$\underline{\underline{\cos(xy) \cdot xy'}} + \underline{\underline{2yy' \cdot \sin(y^2)}} = -\cos(xy) \cdot y$$

$$y'(x\cos(xy) + 2y\sin(y^2)) = -\cos(xy) \cdot y$$

$$y' = \frac{-\cos(xy) \cdot y}{x\cos(xy) + 2y\sin(y^2)}$$

Thus,

$$\left. \frac{dy}{dx} \right|_{(2\pi, \pi)} = \frac{-\cos(2\pi) \cdot \pi}{2\pi \cos(2\pi) + 2\pi \sin(\pi)} = \boxed{-\frac{1}{2}}$$

WS - 14

1, 2

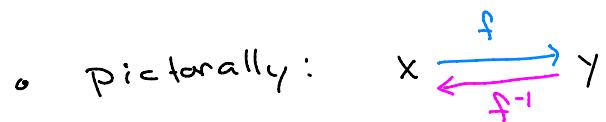
Inverse Trig Functions

Recall:

① Not all functions have an inverse!

② If $f(x)$ has an inverse $f^{-1}(x)$, then

- $f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(y)) = y$



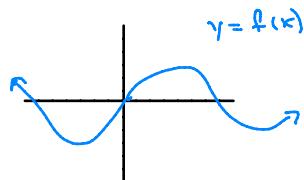
- $f(x) = y \iff f^{-1}(y) = x$

- $(x, y) \text{ is on graph of } f \iff (y, x) \text{ is on graph of } f^{-1}$

③ Be careful: in general, $f^{-1}(x) \neq (f(x))^{-1} = \frac{1}{f(x)}$

Ex Graph f and f^{-1} , if it exists

(a) $f(x) = \sin x$



- f^{-1} DNE b/c $f(x)$

does NOT pass horizontal line test.

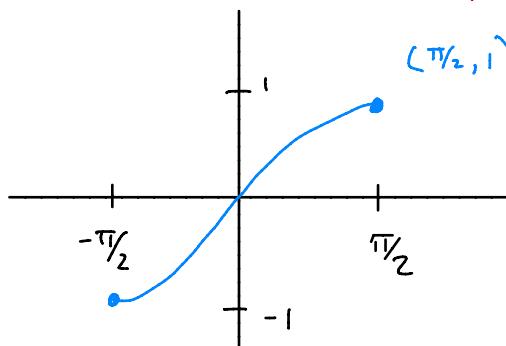
- $f(0) = 0, f(\pi) = 0$

$$0 \rightarrow 0 \\ \pi \nearrow$$

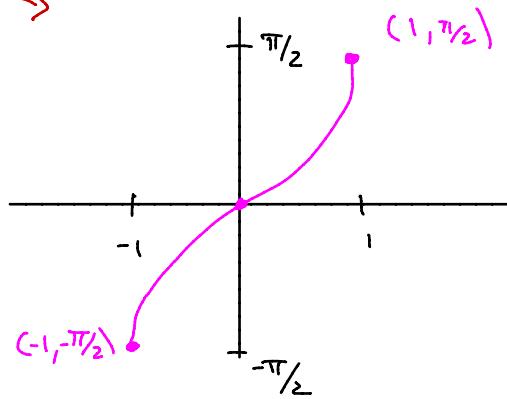
so if f^{-1} exists, $f'(0) = 0, \pi, \dots$
not a function

$$(b) f(x) = \sin x, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

reflect over $y=x$



(x, y) on graph of f



(y, x) on graph of f^{-1}

Def

$\arcsin x = \sin^{-1} x$ is the inverse of $\sin x$ on $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

$\arccos x = \cos^{-1} x$ " " " " " $\cos x$ " $0 \leq x \leq \pi$

$\arctan x = \tan^{-1} x$ " " " " " $\tan x$ " $-\frac{\pi}{2} < x < \frac{\pi}{2}$

(see book for others)

!! $\sin^{-1} x \neq (\sin x)^{-1}$

Let's find $(\sin^{-1} x)'$.

• $y = \sin^{-1}(x) \longleftrightarrow \underline{\sin(y) = x}$ work implicitly!

$$[\sin(y)]' = [x]'$$

$$\cos(y) \cdot y' = 1$$

$$y' = \frac{1}{\cos(y)}$$

can we simplify this?

$\sin y = x$ gives the triangle:



$$(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$$

so, $\cos(y) = \sqrt{1-x^2}$

WS-15

Der. of Inv. Trig

Ex Find the derivative of $f(x) = \sec^{-1}(x^3)$

$$f'(x) = \frac{1}{u\sqrt{u^2-1}} \cdot u' = \boxed{\frac{1}{x^3\sqrt{x^6-1}} \cdot 3x^2}$$

3.6 Derivatives of Logs

Let's find $(\ln x)'$.

$$y = \ln x \longleftrightarrow e^y = x \quad \text{work implicitly}$$

$$(e^y)' = (x)'$$

$$e^y \cdot y' = 1$$

$$y' = \frac{1}{e^y}$$

can we simplify this?

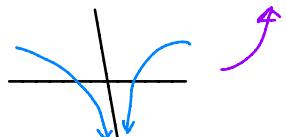
$$e^y = e^{\ln x} = x$$

so

$$\boxed{(\ln x)' = \frac{1}{x}}$$

In fact, this can be extended a bit.

$$\boxed{(\ln|x|)' = \frac{1}{x}}$$



And, in a similar fashion: $(\log_a x)' = \frac{1}{x \ln a}$

WS-15 #1

Logarithmic Differentiation

We know:

$$(x^z)' = zx^{z-1} \quad (\text{power function})$$

$$(z^x)' = z^x \cdot \ln z \quad (\text{exponential func.})$$

$$(x^x)' = ?? \quad (\text{not power nor exponential})$$

Ex Find $(x^*)'$.

Logarithmic Diff

$$y = x^x$$

① Take \ln of both sides & simplify

$$\ln y = \ln(x^x)$$

$$\ln y = x \ln x$$

② Implicit Diff.

$$(\ln y)' = (x \ln x)'$$

$$\frac{1}{y} \cdot y' = x \frac{1}{x} + \ln x$$

$$y' = (1 + \ln x) y$$

③ Sub. back in.

$$y' = (1 + \ln x) x^x$$

Ex Find $\left[(1 + \tan x)^{x^3} \right]'$

$$y = (1 + \tan x)^{x^3}$$

① $\ln y = \ln [(1 + \tan x)^{x^3}]$

$$\ln y = x^3 \ln (1 + \tan x)$$

② $\frac{1}{y} y' = x^3 \cdot \frac{\sec^2 x}{1 + \tan x} + 3x^2 \ln (1 + \tan x)$

$$y' = \left(x^3 \frac{\sec^2 x}{1 + \tan x} + 3x^2 \ln (1 + \tan x) \right) y$$

③ $y' = \left(x^3 \frac{\sec^2 x}{1 + \tan x} + 3x^2 \ln (1 + \tan x) \right) \cdot (1 + \tan x)^{x^3}$

Ex Explain how to proceed.

(a) $(\pi^x)'$ exponential

(b) $((\sin x)^{x+\pi})'$ log.

(c) $((\sin x)^\pi)'$ power + chain

(d) $(x^{\ln x})'$ log.

3.7 (Some Applications)

we just look at velocity and accel.

Recall: If $f(t)$ gives the position of an object at time t , then

position: $f(t)$

velocity: $f'(t)$

acceleration: $f''(t)$

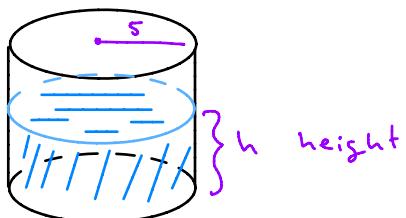
pos. $\xrightarrow{\text{der.}}$ vel. $\xrightarrow{\text{der.}}$ accel.

GW-16 #1, 2

3.9 Related Rates

Ex A large cylindrical tank of radius 5 m is being filled with water. The flow of the water into the tank is measured be $3 \text{ m}^3/\text{min}$. Determine how fast the depth of the water is changing.

① Picture



V is the volume

② Known & Unknown Rates

Known: $\frac{dV}{dt} = 3 \text{ m}^3/\text{min}$

Unknown: $\frac{dh}{dt}$

Q1 what are we trying to find? ... change? ... rate of change? ... change what? change what?

③ Relate Quantities

$$V = \pi r^2 h \rightarrow$$

$$V = 25\pi h$$

Q2 What strategy might we use to solve this?

④ Relate Rates

$$\frac{d}{dt}(V) = \frac{d}{dt}(25\pi h)$$

think $V = V(t)$ $h = h(t)$



$$\frac{dV}{dt} = 25\pi \frac{dh}{dt}$$

⑤ Solve

$$3 = 25\pi \frac{dh}{dt} \rightarrow \frac{dh}{dt} = \frac{3}{25\pi} \text{ m/min} \approx 3.8 \text{ cm/min}$$