4.1 Maximums & Minimums

An important application of calculus is to optimize things, e.g. minimize cost or maximize velocity.

$\boxed{\text{WS-19}}$ Def. Abs. Max/Min, # 1

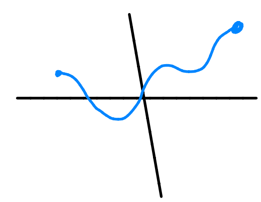
$\boxed{\text{WS-19}}$ # 2, 3, 4, 5

$\boxed{\text{WS-19}}$ Ext. Value Theorem

<u>Question</u> How can we find abs. extrema (if they exist)?

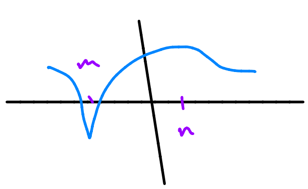
Main idea:

1. look at local extrema
2. look at end points



<u>Finding local extrema</u>

What is true about der. at a local max/min.



local max: $f'(n) = 0$

local min: $f'(m)$ DNE

$\boxed{\text{WS-20}}$ Local Ext. Thm, # 1

optional

<u>Ex</u> Find critical #s of $g(x) = \dfrac{2}{x^2 - x - 2}$

$$g'(x) = \cdots = \frac{2-4x}{(x^2-x-2)^2}$$

<u>$f'(x) = 0$</u>

$x = 1/2$

<u>$f'(x)$ DNE</u>

$x = 2, -1$

not in domain so not a crit. #

Finding Absolute Extrema

$\boxed{\text{WS-20}}$ Strategy

Ex Find the abs. extr. of $f(x) = 2x^3 - 3x^2 - 36x$ on $[0, 10]$

① Find critical #s

from WS20 $\longrightarrow$ $f'(x) = 6x^2 - 6x - 36$

$f'(x) = 0$ oR $f'(x)$ DNE

$x = 3, -2$ — none

not in interval

② Test

x	$f(x)$
0	0
10	1340
3	-81

abs. max of 1340 (when $x = 10$)
abs. min of -81 (when $x = 3$)

$\boxed{\text{WS-20}}$ #2

Ex Find abs. extr. of $f(x) = 3x^{2/3} - x$ on $[-1, 8]$

① Find crit. #s

$f'(x) = 2x^{-1/3} - 1 = \frac{2}{\sqrt[3]{x}} - 1$

$f'(x) = 0$

$\frac{2}{\sqrt[3]{x}} - 1 = 0 \Rightarrow x = 8$

$f'(x)$ DNE

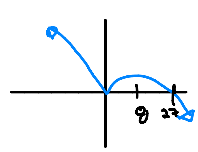
$x = 0$

② Test

x	$f(x)$
-1	4
8	4
0	0

abs max of 4 $(x = -1, 8)$
abs min of 0 $(x = 0)$

Graph Me!



4.2 Mean Value Theorem

WS-21 #1

Rolle's Theorem Suppose that f satisfies

1. f is continuous on $[a,b]$,
2. f is differentiable on (a,b), and
3. $f(a)=f(b)$.

hor. tan. line at $x=c$

Then there is at least one x-value c in (a,b) s.t. $f'(c)=0$.

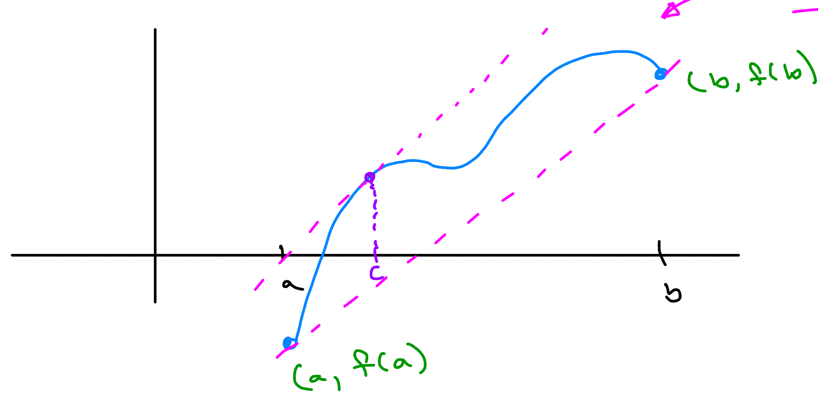
pf

If f is constant, $f'(c)=0$ for all c in (a,b).

If f is not constant, f achieves a max or min different than $f(a)$ in (a,b) by EVT. Assume this happens at $x=c$. By Local Extrema Theorem, $f'(c)=0$ or $f'(c)$ DNE, but second is ruled out by hypothesis ②. $\square$

Question: what if $f(a) \neq f(b)$... can we say anything?

parallel!



W5-21

Mean Value Theorem Suppose that f satisfies

1. f is continuous on $[a,b]$, AND
2. f is differentiable on (a,b).

Then there is at least one x-value c in (a,b) s.t.

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

inst. rate of change

avg. rate of change

"Applied" Example

You are driving to visit a friend and decide to take the tollway because the speed limit is a pleasant 70 miles per hour. When you enter the tollway around 12:00PM, you are given a paper card that you will use to pay when you get off of the tollway. The card records the time and the location where you entered the tollway. After 36 miles, you exit the tollway at 12:30PM. You give your card to the attendant, and you are immediately issued a speeding ticket for \$100. How can they *prove* you were speeding?

pf (idea)

Let $h(x) = \underbrace{f(x) - \left[\frac{f(b)-f(a)}{b-a}(x-a) + f(a)\right]}_{\text{the dist. b/w } f \text{ and secant line}}$.

Now, apply Rolle's Thm to $h(x)$. $\square$

Theorem If $f'(x)=0$ for all x in (a,b), then f is constant on (a,b).

pf

We want to show that if $a<u,v<b$ then $f(u)=f(v)$. Now, f is cont. and diff. on $[u,v]$ so by the MVT, there is a $u<c<v$ s.t.

$$0 = f'(c) = \frac{f(v)-f(u)}{v-u}.$$

Thus, $0 = f(v)-f(u)$, so $f(u)=f(v)$. $\square$

$\boxed{\text{WS-21}}$ "Same Der. Theorem"

WS-21

Theorem If $f'(x) = g'(x)$, then $f(x) = g(x) + C$ for some constant C.

Pf

We want to show $f(x) = g(x) + C$, which is the same as showing $f(x) - g(x) = C$. Since $f'(x) = g'(x)$, $(f(x) - g(x))' = 0$, so by the previous result, $f(x) - g(x) = C$. $\square$

$\boxed{\text{WS-21}}$ #2

4.3 What Does f' & f'' tell us about f

Increasing / Decreasing & Concavity

$\boxed{\text{WS-22}}$ Def. * draw pictures on side

$\boxed{\text{WS-22}}$ #1, 2

$\boxed{\text{WS-22}}$ Theorem: connect incr/decr/concavity with f', f''

$\boxed{\text{WS-22}}$ #3, 4

Finding Increasing/Decreasing, Concavity, Extrema, & IPs

$\boxed{\text{WS-23}}$ Strategy

Ex Let $f(x) = 2x^3 + 3x^2 - 36x$. Find intervals of incr, decr., CU, CD. Also find local extrema + inf. pts.

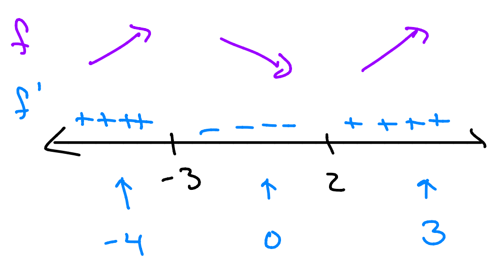
(I) $f'(x) = 6x^2 + 6x - 36 = 6(x^2 + x - 6) = 6(x+3)(x-2)$

$f'(x) = 0$

$x = -3,\ x = 2$

$f'(x)$ DNE

none



$f'(-4) = 36$ +

$f'(0) = -36$ -

$f'(3) = 36$ +

incr.: $(-\infty, -3), (2, \infty)$

decr.: $(-3, 2)$

local max: 81 when $x = -3$

local min: -44 when $x = 2$

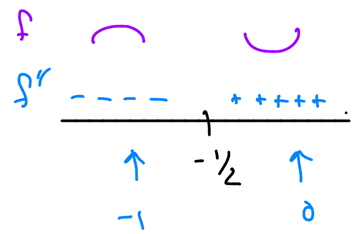
II $\quad f''(x) = 12x + 6 = 6(2x+1)$

$\underline{f''(x) = 0}$

$x = -\frac{1}{2}$

$\underline{f'' \text{ DNE}}$

Never



$f''(-1) < 0 \quad -$

$f''(0) > 0 \quad +$

$$\boxed{\begin{array}{ll} \underline{\text{CU}}: \left(-\tfrac{1}{2}, \infty\right) & \text{I.P. } \left(-\tfrac{1}{2}, -37.5\right) \\ \underline{\text{CD}}: \left(-\infty, -\tfrac{1}{2}\right) & \end{array}}$$

$\underline{\text{Ex}}$ Find all local extrema and inf. pts of

$$f(x) = x^{2/3}(6-x)^{1/3}.$$

Note that

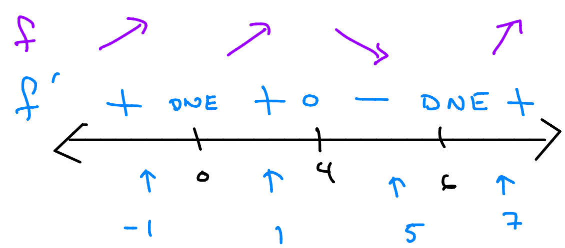
$$f'(x) = \frac{4-x}{x^{1/3}(6-x)^{2/3}} \qquad f''(x) = \frac{-8}{x^{4/3}(6-x)^{5/3}}$$

I

$\underline{f'(x) = 0}$

$4 - x = 0$

$x = 4$



$\underline{f'(x) \text{ DNE}}$

$x^{1/3} = 0$ or $(6-x)^{2/3} = 0$

$x = 0 \qquad x = 6$

$f'(-1) \quad \frac{(+)}{(-)(+)} = +$

$f'(1) \quad \frac{(+)}{(+)(+)} = +$

$f'(5 \quad \frac{(-)}{(+)(+)} = -$

$f'(7) \quad \frac{(-)}{(+)(+)} = +$

$$\boxed{\begin{array}{l} \text{local max of } f(4) = 2^{5/3} \approx 3.2 \text{ when } x = 4 \\ \text{local min of } f(6) = 0 \text{ when } x = 6 \end{array}}$$

II

$f''(x) = 0$	$f''(x)$ DNE
never	$x = 0, 6$

	$x<0$	$0<x<6$	$x>6$
f	⌢	⌢	⌣
f''	$-$	$-$	$+$
test	-1	1	7

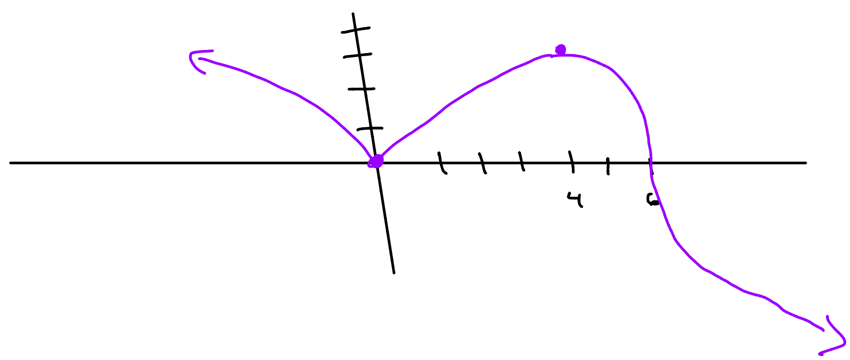
$f''(-1)$ $\dfrac{(-)}{(+)(+)} = -$

$f''(1)$ $\dfrac{(-)}{(+)(+)} = -$

$f''(7)$ $\dfrac{(-)}{(+)(-)} = +$

$\boxed{\text{One I.P.}: \underbrace{(6,0)}_{f(6)}}$

For fun:



$\boxed{\text{GW-23}}$ # 1

4.4 L'Hôpital's Rule

Recall:

① $\lim_{x\to 2} \frac{x^2-4}{x-2} \overset{\frac{0}{0}}{} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}} = \lim_{x\to 2} (x+2) = 4$

② $\lim_{x\to\infty} \frac{3x^2-x}{-x^2+4x-1} \overset{\frac{\infty}{\infty}}{} = \lim_{x\to\infty} \frac{3x^2-x}{-x^2+4x-1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{3-\cancel{\frac{1}{x}}^{\nearrow 0}}{-1+\cancel{\frac{4}{x}}^{\nearrow 0}-\cancel{\frac{1}{x^2}}^{\nearrow 0}} = -3$

But, what about...

③ $\lim_{x\to 1} \frac{\ln x}{x-1} \overset{\frac{0}{0}}{}$ OR $\lim_{x\to\infty} \frac{\ln x}{x-1} \overset{\frac{\infty}{\infty}}{}$

$\frac{0}{0}$ and $\frac{\infty}{\infty}$ ← includes $\frac{\pm\infty}{\pm\infty}$ represent limits and are called <u>indeterminate forms.</u>

GW-24 L'Hôpital's Rule

<u>Ex</u> compute

① $\lim_{x\to 1} \frac{\ln x}{x-1} \overset{\frac{0}{0}}{} \underset{HR}{=} \lim_{x\to 1} \frac{\frac{1}{x}}{1} = \boxed{1}$

② $\lim_{x\to\infty} \frac{\ln x}{x-1} \overset{\frac{\infty}{\infty}}{} \underset{HR}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \boxed{0}$

③ $\lim_{x\to\infty} \frac{x+\sin x}{x} \overset{\frac{\infty}{\infty}}{} \underset{HR}{=} \lim_{x\to\infty} \frac{1+\cos x}{1}$ <u>DNE</u> so start over!!

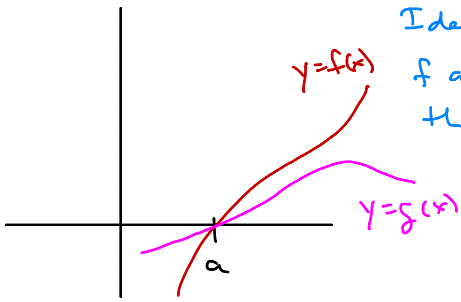
HR does <u>NOT</u> apply

$\lim_{x\to\infty} \frac{x+\sin x}{x} = \lim_{x\to\infty} 1 + \cancel{\frac{\sin x}{x}}^{\nearrow 0} = \boxed{1}$

④ $$\lim_{t\to 0} \frac{\sqrt{1+t} - 1 - t/2}{t^2} \overset{\frac{0}{0}}{\underset{HR}{=}} \lim_{t\to 0} \frac{\frac{1}{2}(1+t)^{-1/2} - \frac{1}{2}}{2t} \overset{\frac{0}{0}}{\underset{HR}{=}} \lim_{t\to 0} \frac{-\frac{1}{4}(1+t)^{-3/2}}{2} = \boxed{-\frac{1}{8}}$$

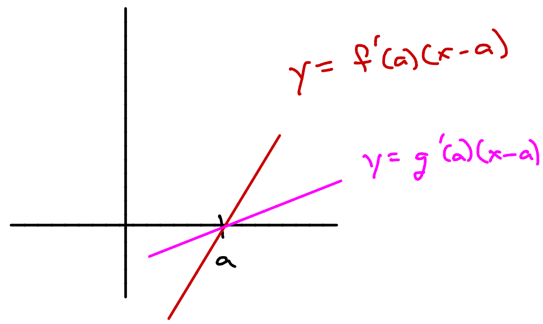
pf of HR (idea for $\frac{0}{0}$)

* assume f' and g' are continuous



Idea: near a
f and g look like
their tangent lines
so

$$\frac{f(x)}{g(x)} \approx \frac{f'(a)(x-a)}{g'(a)(x-a)}$$



Now,
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(a)(x-a)}{g'(a)(x-a)} = \lim_{x\to a} \frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

f', g' continuous

Indeterminate Products

What about $\lim_{x\to 0^+} x\ln x$? $0\cdot(-\infty)$? ... indetermiate!

includes $\pm\infty$

* $0\cdot\infty$ is an indetermiate form!

Strategy

If $\lim_{x\to a} f(x)g(x)$ has the form $0\cdot\infty$, rewrite limit as

$$\lim_{x\to a} \frac{f(x)}{(g(x))^{-1}} \quad \text{OR} \quad \lim_{x\to a} \frac{g(x)}{(f(x))^{-1}}$$

Ex compute

① $$\lim_{x\to 0^+} x\ln x \overset{0\cdot(-\infty)}{=} \lim_{x\to 0^+} \frac{\ln x}{x^{-1}} \overset{\frac{-\infty}{\infty}}{\underset{HR}{=}} \lim_{x\to 0^+} \frac{x^{-1}}{-x^{-2}} = \lim_{x\to 0^+} -x = \boxed{0}$$

$0\cdot\infty$

(2) $\displaystyle\lim_{x\to\infty} x^{-3}e^{x^2} = \lim_{x\to\infty} \frac{e^{x^2}}{x^3} \overset{\frac{\infty}{\infty}}{\underset{HR}{=}} \lim_{x\to\infty} \frac{e^{x^2}\cdot 2x}{3x^2}$

$\displaystyle = \lim_{x\to\infty} \frac{2e^{x^2}}{3x} \overset{\frac{\infty}{\infty}}{\underset{HR}{=}} \lim_{x\to\infty} \frac{2e^{x^2}\cdot 2x}{3} = \boxed{\infty}$

optional

(3) $\displaystyle\lim_{x\to\infty} x\ln\left(\frac{x-1}{x+1}\right)$

Indeterminate Differences

What about $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{e^x-1}\right)$? $\infty-\infty$

* $\infty-\infty$ is an indeterminate form... but not $\infty+\infty$, why?

Strategy

combine with common denominator

Ex Compute

$\displaystyle\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{e^x-1}\right) = \lim_{x\to 0^+} \frac{e^x-1-x}{x(e^x-1)}$ $\frac{0}{0}$

$\displaystyle \underset{HR}{=} \lim_{x\to 0^+} \frac{e^x-1}{xe^x+e^x-1}$ $\frac{0}{0}$

$\displaystyle \underset{HR}{=} \lim_{x\to 0^+} \frac{e^x}{xe^x+e^x+e^x}$

$= \boxed{\frac{1}{2}}$

GW-24 #1

Indeteminate Powers

what about $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$? 1^∞ ... indeterminate.

* 1^∞, 0^0, ∞^0 are indeterminate ... but ∞^∞ is not, why?

Strategy

use logarithms. let's see this by example.

Ex Compute

① $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$

① Take ln

$$\lim_{x\to\infty} \ln\left(\left(1+\frac{1}{x}\right)^x\right) = \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) \quad \infty\cdot 0$$

$$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{1/x} \quad \frac{0}{0}$$

$$\underset{HR}{=} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\cdot\cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{\left(-\frac{1}{x^2}\right)}}$$

$$= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1$$

② Exponentiate

$$\lim_{x\to\infty} \ln\left(\left(1+\frac{1}{x}\right)^x\right) = 1 \Rightarrow \lim_{x\to\infty} e^{\ln\left(\left(1+\frac{1}{x}\right)^x\right)} = e^1$$

$$\Rightarrow \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = \boxed{e}$$

optional

(2) $\lim_{x \to 0^+} x^x$

GW-24 #2

4.5 Curve Sketching

Ex Find the horizontal asymptotes of $f(x) = \sqrt{x}\, e^{-x}$

* need to find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$

$$\circ \lim_{x\to\infty} \sqrt{x}\, e^{-x} \overset{\infty\cdot 0}{=} \lim_{x\to\infty} \frac{\sqrt{x}}{e^x} \overset{\frac{\infty}{\infty}}{\underset{HR}{=}} \lim_{x\to\infty} \frac{\frac{1}{2}x^{-1/2}}{e^x} = \lim_{x\to\infty} \frac{1}{2\sqrt{x}\, e^x} = \boxed{0}$$

$$\circ \lim_{x\to-\infty} \sqrt{x}\, e^{-x}$$! not defined for $x<0$ so DNE

Thus, there is on horizontal asymptote : $\boxed{y=0}$

Ex Sketch the graph of $f(x) = \sqrt{x}\, e^{-x}$

Domain : $x \geq 0$

Intercepts :

y-int : $y = 0e^0 \implies y = 0$

x-int : $0 = \sqrt{x}\, e^{-x} \implies x = 0$

Asymptotes :

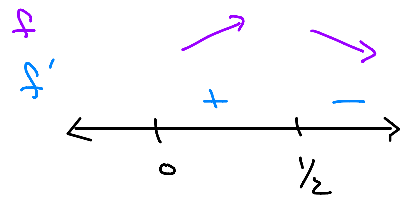
vert : none

hor : check $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$

$\boxed{y=0}$ see previous example.

Inc. / Dec. / Local Extrema

Note that: $f'(x) = e^{-x}\left(\frac{1-2x}{2\sqrt{x}}\right)$



$f'(x)=0$	$f'(x)$ DNE
$x = 1/2$	$x \leq 0$

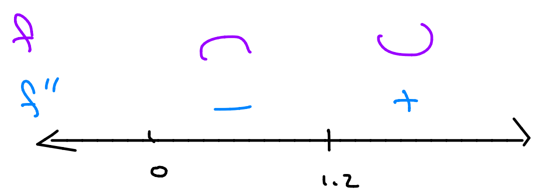
local max: $\left(\frac{1}{2}, f\left(\frac{1}{2}\right)\right) \approx \left(\frac{1}{2}, 0.43\right)$

Concavity / IP

Note that $f''(x) = e^{-x}\left(\frac{-1-4x+4x^2}{4x^{3/2}}\right)$

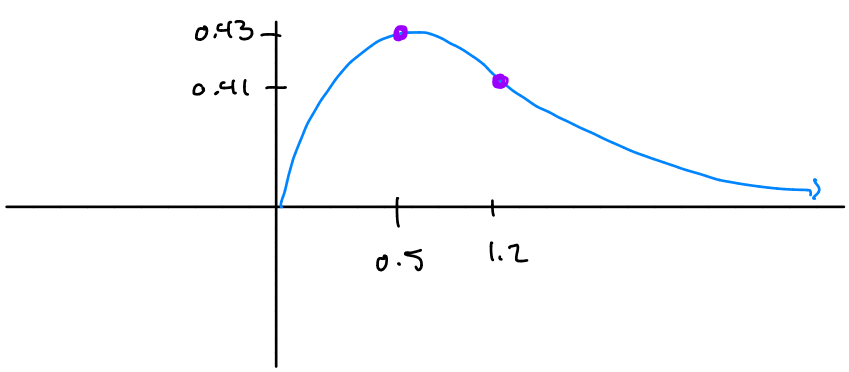
$f''(x)=0$	$f''(x)$ DNE
$x = \frac{1 \pm \sqrt{2}}{2}$	$x \leq 0$
$\approx -0.2, 1.2$	

← not in domain



I.P. $\approx (1.2, 0.41)$

The Sketch



optional

Ex Sketch the graph of $y = \frac{(x-4)^2}{x^2-4}$.

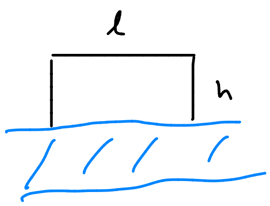
Note: $f'(x) = \frac{8(x-4)(x-1)}{(x^2-4)^2}$

$$f''(x) = -\frac{8(2x^3-15x^2+24x-20)}{(x^2-4)^3}$$

4.7 Optimization

Ex A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river; no fence is needed along the river. What are the dimensions of the field that has the largest area?

① Picture



A is area

② what to optimize? constraints?

Take 1 { maximize Area; constraints amount of fencing, dimensions non-neg.

maximize $A = l \cdot h$

constraints $l + 2h = 2400$; $l, h \geq 0$

③ Convert to function of one variable

$l = 2400 - 2h \implies A = (2400-2h)h$

$$\implies \boxed{A(h) = 2400h - 2h^2, \quad 0 \leq h \leq 1200}$$

$l = 0$

* Note: A = 0 at extremes

④ optimize

$A'(h) = 2400 - 4h$

$A'(h) = 0$

$2400 = 4h$

$h = 600$

$A'(h)$ DNE

never

	h	A(h)
c# →	600	A(600) = 720000
ep	0	A(0) = 0
ep	1200	A(1200) = 0

Max area when

$h = 600$

$l = 1200$

WS-25 #1, #2 (see next page)

Author 1 ______________

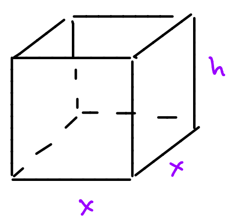
Author 2 ______________

25 – Optimization

Author 3 ______________

1. A large storage crate, with an open top, is to be constructed. The base needs to be a square. Material for the base costs \$10 per square meter, and material for the sides costs \$6 per square meter. If there is \$100 available to spend on the crate, what is the greatest volume of crate that can be built?

① Picture



V is volume

② what to optimize? constraints?

Take! maximize Volume

constraints cost, lengths are non negative

maximize

$$V = x^2 \cdot h$$

constraints

$$100 = 10 \cdot x^2 + 4 \cdot 6 \cdot xh$$

* $100 = 10x^2 + 24xh$

* Also, $x, h \geq 0$

③ Convert to function of one variable

$$\frac{100 - 10x^2}{24x} = h \implies V = x^2 \frac{100 - 10x^2}{24x} = \frac{x}{24}\left(100 - 10x^2\right)$$

$$\implies \boxed{V(x) = \frac{100}{24} \cdot x - \frac{10}{24}x^3 \quad , \quad 0 \leq x \leq \sqrt{10}}$$

$h=0$

*note $V=0$ at both extremes.

④ optimize

$$V'(x) = \frac{100}{24} - \frac{10}{8}x^2$$

$V'(x) = 0$

$$\frac{100}{24} = \frac{10}{8}x^2$$

$$\frac{10}{3} = x^2$$

$$x = \sqrt{\frac{10}{3}}, \cancel{-\sqrt{\frac{10}{3}}}$$

not in the interval

$V'(x)$ DNE

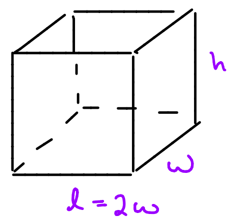
never

	x	$V(x)$
c.# →	$\sqrt{\frac{10}{3}}$	$V\left(\sqrt{\frac{10}{3}}\right) \approx 5.1$
e.p.	0	$V(0) = 0$
	$\sqrt{10}$	$V(\sqrt{10}) = 0$

$$\boxed{\text{max volume is } \approx 5.1\ m^3}$$

2. A large storage crate, with an open top, is to be constructed. The length of the base needs to be twice the width of the base and the volume must be $10\,\text{m}^3$. Material for the base costs \$10 per square meter, and material for the sides costs \$6 per square meter. What is the cost of the materials for the cheapest such container.

① Picture



V is volume

② what to optimize? constraints?

Take! { minimize cost; constraints volume, dimensions }

minimize

$$C = 10(2w^2) + 2\cdot 6\cdot wh + 2\cdot 6\cdot 2wh$$

$$* \quad C = 20w^2 + 36wh$$

constraints

$$* \quad 10 = 2w^2 h$$

$* \quad$ Also, $w, h > 0$ ← not $w, h \geq 0$ b/c of fixed volume!

③ Convert to function of one variable

$$h = \frac{5}{w^2} \Rightarrow C = 20w^2 + 36w\left(\frac{5}{w^2}\right) \Rightarrow \boxed{C(w) = 20w^2 + \frac{180}{w}, \quad 0 < w < \infty}$$

④ optimize

$$C'(w) = 40w - \frac{180}{w^2}$$

$C'(w) = 0$

$$40w = \frac{180}{w^2}$$

$$w^3 = \frac{9}{2}$$

$$w = \sqrt[3]{9/2} \approx 1.65$$

$C'(w)$ DNE

$\cancel{w = 0}$ not in domain

	w	$C(w)$
c.# →	$\sqrt[3]{\frac{9}{2}}$	$C\left(\sqrt[3]{\frac{9}{2}}\right) \approx 163.54$
e.p.	0	$\lim_{w \to 0} C(w) = 0 + \infty = \infty$
	∞	$\lim_{w \to \infty} C(w) = \infty + 0 = \infty$

we haven't seen this before... use limits

$$\boxed{\text{min cost is} \approx \$163.54}$$