5.1 Area & Distance

Q: what is the area of a circle of radius 3cm? ... units?

$9\pi \text{ cm}^2$ ← ❗ square units

... area is defined in terms of squares ...or rectangles.

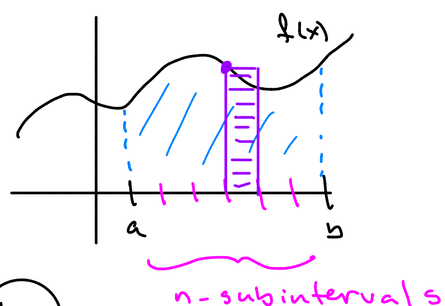
... so how do we find (or define!) the area of other shapes?

$\boxed{\text{WS-26}}$ #1 (see next page)

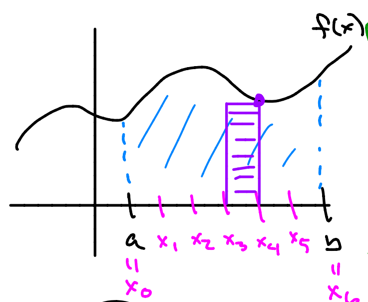
L_n, R_n, and M_n

These are specific ways to estimate area using <u>n rectangles of equal width</u>. The difference is in how the height is chosen.

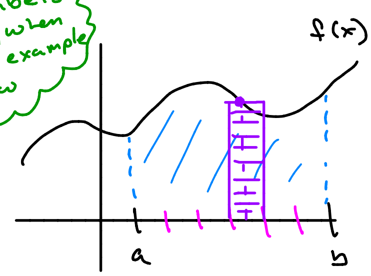
To Josh: add labels later, when doing example below



L_n



R_n



M_n

<u>width of each rectangle</u> $\Delta x = \frac{b-a}{n}$ <u>width</u> $\Delta x = \frac{b-a}{n}$ <u>width</u> $\Delta x = \frac{b-a}{n}$

<u>height</u> use left-hand ep. <u>height</u> use right-hand ep <u>height</u> use midpoint

<u>EX</u> In the middle picture above, there are 6 rect., so

$$\text{Area} \approx R_6 = f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_5)\Delta x + f(x_6)\Delta x$$
$$= \left(f(x_1) + \dots + f(x_6)\right)\frac{b-a}{n}$$

$\boxed{\text{WS-26}}$ #2 (see next page)

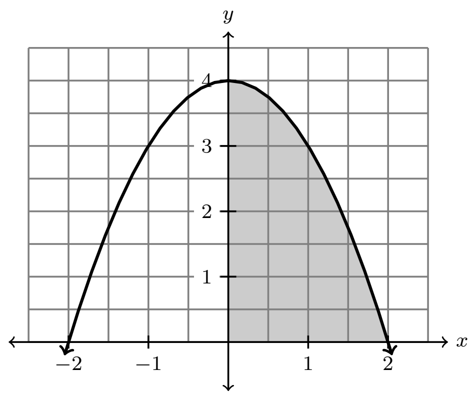
Author 1 ______________________

Author 2 ______________________

Author 3 ______________________

26 – Area

1. The graph of $f(x) = 4 - x^2$ is below. Let A be the area under $f(x)$ from $x = 0$ to $x = 2$.

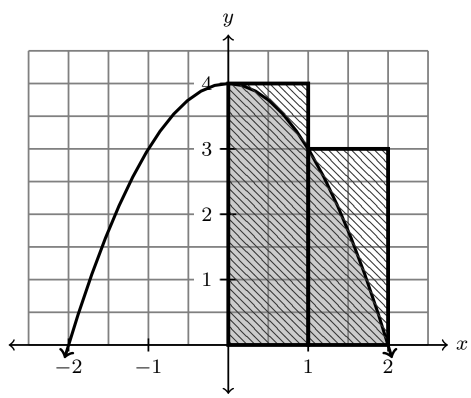


(a) Give your best estimate of the area A that you can. *Make sure to explain your answer.*

Be creative, but explain.

(b) Was your estimate an over or underestimate (or are you not sure)?

Try to figure it out.

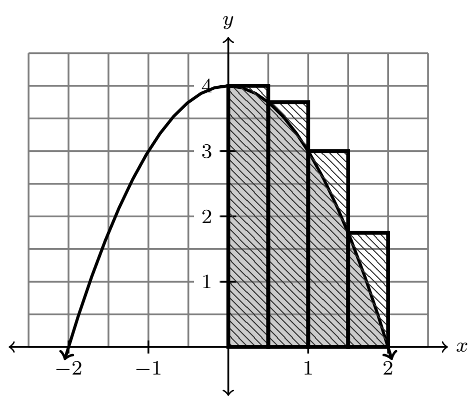


(c) Compute the area in the 2 large rectangles. This estimate of A is called L_2.

$L_2 = h_1 \cdot w + h_2 \cdot w = 4 \cdot 1 + 3 \cdot 1 = 7$

(d) Is L_2 an over or underestimate (or not sure)? *Why?*

over — too much area

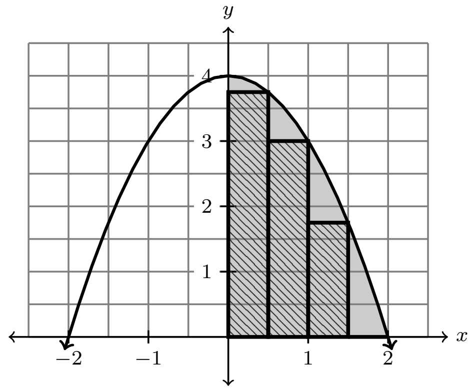


(e) Compute the area in the 4 large rectangles. This estimate of A is called L_4.

$L_4 = 4 \cdot 0.5 + 3.75 \cdot 0.5 + 3 \cdot 0.5 + 1.75 \cdot 0.5$
$= 6.25$

(f) Is L_4 an over or underestimate (or not sure)? *Why?*

over – too much area

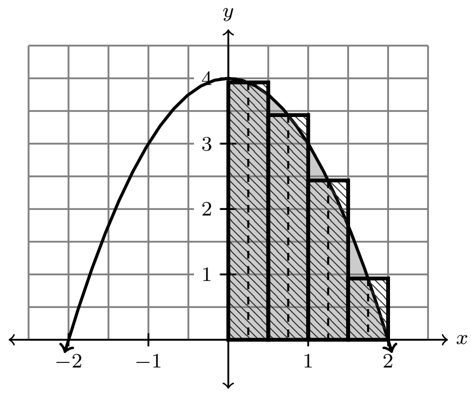


(g) Repeat for these 4 rectangles. This is R_4. (Do you see where the 4^{th} one is?)

$R_4 = (3.75 + 3 + 1.75 + 0) \cdot 0.5 = 4.25$

(h) Is R_4 an over or underestimate (or not sure)?

under



(i) Repeat for these 4 rectangles. This is M_4. To find the heights, use the fact that $f(x) = 4 - x^2$.

$M_4 = \left[(4 - .25^2) + (4 - .75^2) + (4 - 1.25^2) + (4 - 1.75^2)\right] \cdot 0.5$
$= 5.375$

(j) Is M_4 an over or underestimate (or not sure)?

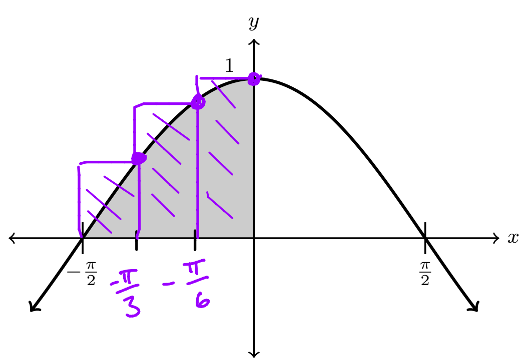
hard to tell... imagine rotating tops of rectangles into tangent lines. ... same area, and now, clearly, over est.

(i) Which do you think is the best estimate of A? How could you get a better estimate?

probably M_4 ... but maybe the one you had in (a).

* final guesses on actual area? ... it ... is ... $5\frac{1}{3}$

2. The graph of $f(x) = \cos x$ is below. Let A be the area under $\cos(x)$ from $x = -\frac{\pi}{2}$ to $x = 0$.



(a) Estimate A using R_3, and **draw the associated rectangles**.

$R_3 = \left(\cos(-\pi/2) + \cos(-\pi/3) + \cos(-\pi/6)\right) \cdot \frac{\pi}{6}$

$= \left(1 + \frac{\sqrt{3}}{2} + \frac{1}{2}\right) \cdot \frac{\pi}{6} = \boxed{\frac{(3+\sqrt{3})\pi}{12}} \approx 1.24$

(b) Is R_3 an over or underestimate (or not sure)?

over!

* any guess for the actual area?
... maybe something with π?
... it ... is ... 1

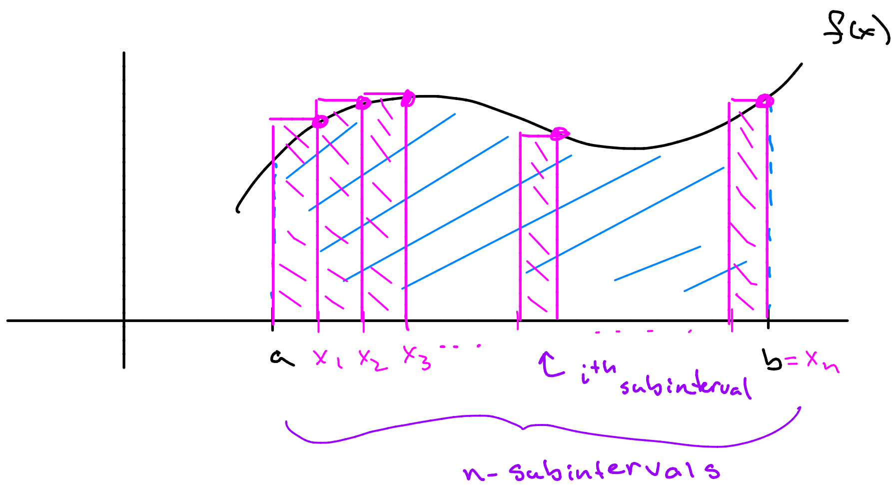
Definition of Area

In the previous problem, did the area actually equal R_3? Would it have equaled R_{10}? R_{1000}?

<u>Def</u> Let f be any continuous function that is nonnegative on $[a,b]$. The <u>area</u> A of the region between f and the x-axis, from $x=a$ to $x=b$, is

$$\boxed{A = \lim_{n\to\infty} R_n} = \lim_{n\to\infty} \left(f(x_1) + \cdots + f(x_n)\right)\Delta x$$

where x_i is the <u>right</u>-hand ep. of i^{th} subinterval and $\Delta x = \frac{b-a}{n}$



* Also, $A = \lim_{n\to\infty} L_n$, $A = \lim_{n\to\infty} M_n$, and others too!

Summation Notation

By example...

end

add up

$$\sum_{i=2}^{7} i^2 = 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^3$$

start

Ex Expand each of the following and simplify.

(a) $\sum_{i=-1}^{1} \cos\left(\frac{\pi \cdot i}{4}\right) = \cos\left(-\frac{\pi}{4}\right) + \cos(0) + \cos\left(\frac{\pi}{4}\right) = \boxed{1+\sqrt{2}}$

(b) $\sum_{i=0}^{3} \frac{2x}{1+(i-1)^2} = \frac{2x}{2} + \frac{2x}{1} + \frac{2x}{2} + \frac{2x}{1+4} = 4x + \frac{2x}{5} = \boxed{\frac{22x}{5}}$

Ex write in summation notation.

$2x^3 + 3x^4 + 4x^5 + 5x^6$

... think ... many possibilities like $\sum_{i=2}^{5} i x^{i+1}$

Back to area

So, returning to our area definition.

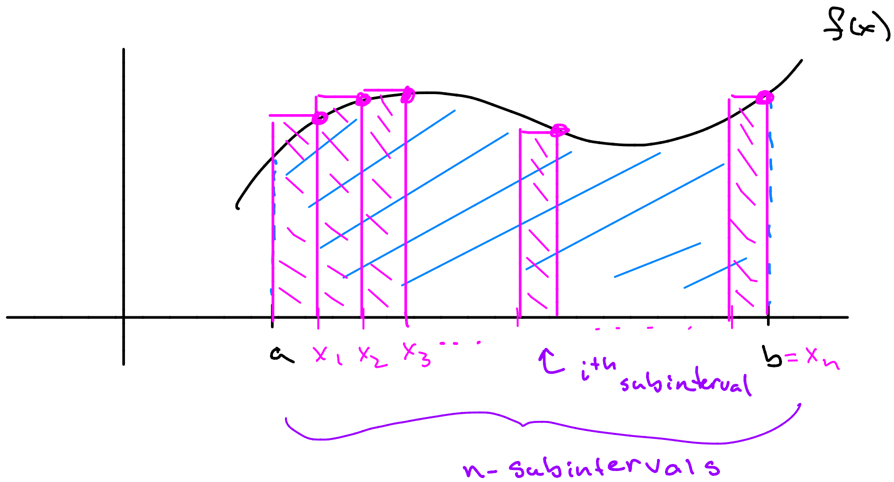
$$A = \lim_{x\to\infty} R_n$$

$$= \lim_{x\to\infty} \left[f(x_1) + f(x_2) + \dots + f(x_n)\right] \Delta x$$

$$= \lim_{x\to\infty} \sum_{i=1}^{n} f(x_i)\, \Delta x$$

5.2 Definite Integral

Recall:



$$A = \lim_{n\to\infty} R_n = \underline{\lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i)\Delta x\right)}$$

this is important! let's give it a name.

* where x_i is right-hand ep. of i^{th} sub. int.

and $\Delta x = \frac{b-a}{n}$

Def Suppose f is defined on $[a,b]$. If

$$\lim_{x\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

exists and is always the same for all possible choices of sample points x_i (i.e. $R_n, L_n, M_n, \ldots$), the we say f is integrable on $[a,b]$, and we write

$$\int_a^b f(x)\,dx = \lim_{x\to\infty} \underbrace{\sum_{i=1}^{n} f(x_i)\,\Delta x}_{\text{Riemann sum}}$$

integral sign

upper limit

lower limit

integrand

Theorem If f is continuous or has only a finite number of jump discontinuities on $[a,b]$, then

$$\int_a^b f(x)\,dx \text{ exists}$$

i.e. f is integrable on $[a,b]$.

Computing $\int_a^b f(x)\,dx$ Geometrically

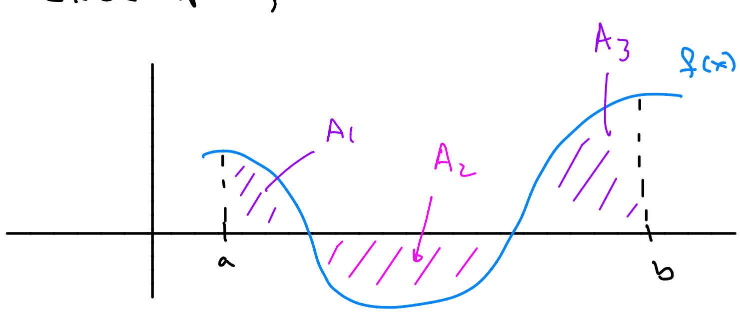
* $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$ which is how we defined the area under f ... when f was positive.

* If f becomes negative, we get "negative area"

Theorem $\int_a^b f(x)\,dx$ is the net area between f and the x-axis. That is

$$\int_a^b f(x)\,dx = \begin{pmatrix}\text{total area}\\ \text{above x-axis}\end{pmatrix} - \begin{pmatrix}\text{total area}\\ \text{below x-axis}\end{pmatrix}$$

For example, if...



$$\int_a^b f(x)\,dx = A_1 + A_3 - A_2$$

$\boxed{\text{WS-27}}$ #1, 2, 3 (see next page)

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AUTHOR 2 ______________________

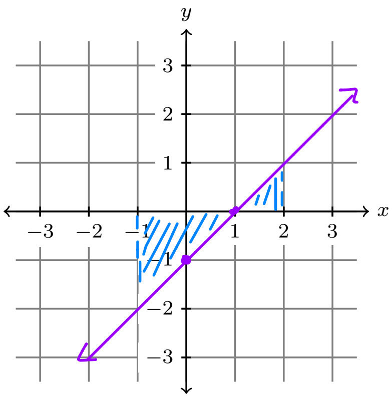
27 – Definite Integral

AUTHOR 3 ______________________

Theorem: Evaluating Definite Integrals Geometrically

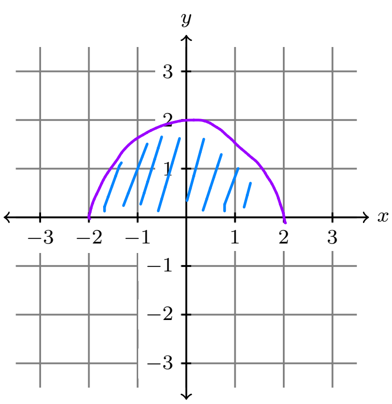
$$\int_a^b f(x)\,dx = \begin{pmatrix}\text{total area under } f \text{ and} \\ \textit{above} \text{ the } x\text{-axis}\end{pmatrix} - \begin{pmatrix}\text{total area above } f \text{ and} \\ \textit{below} \text{ the } x\text{-axis}\end{pmatrix}$$

1. Graph $f(x) = x - 1$ over $[-1, 2]$, and evaluate $\int_{-1}^{2} (x-1)\,dx$ by interpreting it as (net) area.



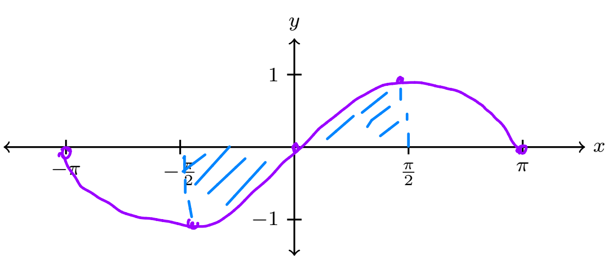
$$\int_{-1}^{2} (x-1)\,dx = \frac{1}{2} - 2 = \boxed{-\frac{3}{2}}$$

2. Graph $f(x) = \sqrt{4 - x^2}$ over $[-2, 2]$, and evaluate $\int_{-2}^{2} \sqrt{4-x^2}\,dx$ by interpreting it as (net) area.



$$\int_{-2}^{2} \sqrt{4-x^2}\,dx = \frac{1}{2} \cdot \pi(2^2) = \boxed{2\pi}$$

3. Graph $f(x) = \sin x$ over $[-\frac{\pi}{2}, \frac{\pi}{2}]$, and evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x\,dx$ by interpreting it as (net) area.



$$\int_{-\pi/2}^{\pi/2} \sin x\,dx = \boxed{0}$$

* but, what about $\int_0^{\pi/2} \sin x\,dx$?

... much harder!

Properties of the Definite Integral

Notational

(a) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ why: LHS $\Delta x = \frac{b-a}{n}$ RHS $\Delta x = \frac{a-b}{n}$

(b) $\int_a^a f(x)\,dx = 0$ why: 0 area

Algebraic

① $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$

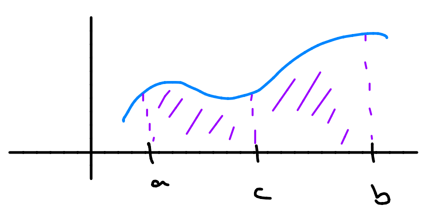
why: integrand is c-times larger area is c-times larger

② ③ $\int_a^b \left(f(x) \pm g(x)\right) dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

why: area "under" f stacked on top of area "under" g

This one is very useful! ④ $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

why



⑤ If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

"we can integrate inequalities"

Ex If $\int_{-3}^7 f(x)\,dx = 4$ and $\int_1^7 f(x)\,dx = 9$, find $\int_{-3}^1 7f(x)\,dx$.

$$\underbrace{\int_{-3}^7 f(x)\,dx}_{4} = \int_{-3}^1 f(x)\,dx + \underbrace{\int_1^7 f(x)\,dx}_{9} \Rightarrow \int_{-3}^1 f(x)\,dx = -5$$

$$\Rightarrow \int_{-3}^1 7\cdot f(x)\,dx = 7\int_{-3}^1 f(x)\,dx = \boxed{-35}$$

Computing $\int_a^b f(x)\,dx$ Algebraically

Suppose f is integrable on $[a,b]$.

Remember: this is implied by continuity

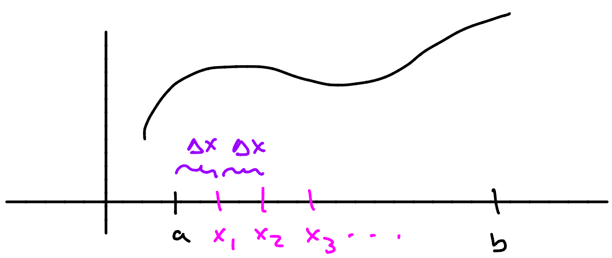
Then, by our definition

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} R_n \quad (\text{or } L_n, M_n \ldots)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

* where x_i is the right-hand endpt. of i^{th} subinterval and $\Delta x = \frac{b-a}{n}$

Let's develop a formula for x_i



$x_1 = a + \Delta x$

$x_2 = a + 2\Delta x$

$x_3 = a + 3\Delta x$

$\vdots$

$x_i = a + i\Delta x$

<u>Thm</u> If f is integrable, then

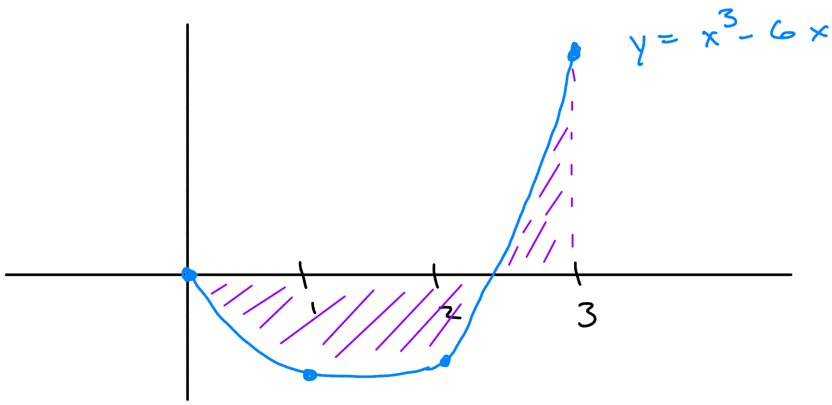
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,dx$$

where

- $\Delta x = \dfrac{b-a}{n}$
- $x_i = a + i\left(\dfrac{b-a}{n}\right)$

Ex Consider $\int_0^3 (x^3 - 6x)\,dx$.

(a) Draw the area represented by the integral.



(b) Estimate the integral using R_3

$\frac{b-a}{n}$

$$\int_0^3 (x^3 - 6x)\,dx \approx R_3 = \left(f(x_1) + f(x_2) + f(x_3)\right)\Delta x$$
$$= \left(f(1) + f(2) + f(3)\right) \cdot 1$$
$$= (-5 - 4 + 9)$$
$$= \boxed{0}$$

← hmm... that doesn't seem like a very good estimate

(c) Write the integral as a limit of Riemann sums.

$$\int_0^3 (x^3 - 6x)\,dx = \lim_{n\to\infty} R_n$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$\Delta x = \frac{b-a}{n} = \frac{3}{n}$

$x_i = a + i\Delta x$
$= 0 + i \cdot \frac{3}{n}$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right) \cdot \frac{3}{n}$$

$$= \boxed{\lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right] \cdot \frac{3}{n}}$$

(d) compute the integral!

$$\int_0^3 (x^3-6x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n}\left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right]\cdot\frac{3}{n}$$

Let's first simplify the sum.

$$\sum_{i=1}^{n}\left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right]\cdot\frac{3}{n} = \sum_{i=1}^{n}\left[\frac{27i^3}{n^3} - \frac{18i}{n}\right]\cdot\frac{3}{n}$$

$$= \sum_{i=1}^{n}\left[\frac{81}{n^4}i^3 - \frac{54}{n^2}i\right]$$

see below

$$= \sum_{i=1}^{n}\frac{81}{n^4}i^3 - \sum_{i=1}^{n}\frac{54}{n^2}i$$

see below

$$= \frac{81}{n^4}\sum_{i=1}^{n} i^3 - \frac{54}{n^2}\sum_{i=1}^{n} i$$

$$= \frac{81}{n^4}\left[\frac{n(n+1)}{2}\right]^2 - \frac{54}{n^2}\left[\frac{n(n+1)}{2}\right]$$

$$= \frac{81}{2}\frac{(n+1)^2}{n^2} - 27\left(\frac{n+1}{n}\right)$$

$$\sum_{i=1}^{n}(a_i-b_i) = (a_1-b_1)+(a_2-b_2)+\cdots+(a_n-b_n)$$
$$= (a_1+a_2+\cdots+a_n) - (b_1+b_2+\cdots+b_n)$$
$$= \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} ca_i = ca_1+ca_2+\cdots+ca_n$$
$$= c(a_1+a_2+\cdots+a_n)$$
$$= c\sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n} i = 1+2+\cdots+n = (\text{think}) = \frac{n(n+1)}{2}$$

hmm... Think like Gauss

$$\begin{array}{rl} & 1+2+\cdots+(n-1)+n \\ + & n+(n-1)+\cdots+2+1 \\ \hline & (n+1)+\cdots\quad+(n+1)+(n+1) \end{array} \implies 2(1+2+\cdots+n) = n\cdot(n+1)$$

Also, from book

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Now we take the limit.

OR use HR

$$\lim_{n\to\infty}\left[\frac{81}{2}\frac{(n+1)^2}{n^2}-27\left(\frac{n+1}{n}\right)\right]=\lim_{n\to\infty}\left[\frac{81}{2}\left(1+\cancel{\tfrac{1}{n}}^{\,0}\right)^2-27\left(1+\cancel{\tfrac{1}{n}}^{\,0}\right)\right]$$

$$=\boxed{\frac{81}{2}-27}=\boxed{-6.75}$$

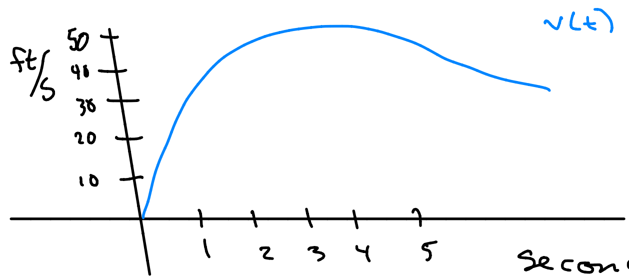
YAY!!

$\boxed{\text{WS-27}}$ #4 (see two pages ahead in notes)

Distance from velocity

Recall that for constant velocity: distance = (velocity)·(time)

Now suppose velocity is NOT constant. Suppose a car is moving with a velocity of $v(t)$.



How could we determine the distance traveled from time 0 to time 5 min?

* we could pretend the velocity was constant, say 40 ft/s

Then,

$$\text{dis.} \approx 40\cdot 5 = 200 \text{ ft}$$

* better, we could just pretend the velocity was constant over each one second interval

$$\text{dis.} \approx v(1)\cdot 1 + v(2)1 + v(3)\cdot 1 + v(4)\cdot 1 + v(5)\cdot 1$$
$$\approx 20 + 45 + 52 + 52 + 50$$
$$\approx 219 \text{ ft}$$

* even better, we could work with smaller time intervals

$$\text{dis.} \approx [v(t_1) + v(t_2) + \cdots + v(t_n)]\,\Delta t$$ ← for Δt small

This leads to a theorem.

this accounts for negative velocity – think of a spring.

<u>Theorem</u> The (net) dispacement, D, of an object moving with velocity $v(t)$ from time $t=a$ to $t=b$ is

$$D = \int_a^b v(t)\,dt = \lim_{n\to\infty} R_n$$

Summary

① $\int_a^b f(x)dx$ computes the net area b/w f and the x-axis from $x=a$ to $x=b$.

② If $f(x)$ is velocity at time x, then $\int_a^b f(x)dx$ gives displacement.

$\boxed{\text{WS-27}}$ #5 (see next page)

Theorem: Evaluating Definite Integrals Algebraically

If f is integrable, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i)\Delta x\right)$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\left(\frac{b-a}{n}\right)$.

4. Consider the integral $\displaystyle\int_1^3 \frac{1}{1+x^2}\,dx$.

(a) Estimate the integral using R_4 (4 subintervals with right-hand endpoints as sample points).

1 1.5 2 2.5 3

$\Delta x = \frac{2}{4} = \frac{1}{2}$

$$R_4 = \left[\frac{1}{1+(1.5)^2} + \frac{1}{1+(2)^2} + \frac{1}{1+(2.5)^2} + \frac{1}{1+(3)^2}\right]\cdot\frac{1}{2}$$

$\approx \boxed{0.37}$

(b) Express the integral as a limit of Riemann sums. (But, do not compute it.)

$$\int_1^3 \frac{1}{1+x^2}\,dx = \boxed{\lim_{n\to\infty}\left(\sum_{i=1}^{n}\left(\frac{1}{1+(1+\frac{i}{2})^2}\cdot\frac{1}{2}\right)\right)}$$

Theorem: Evaluating Definite Integrals Algebraically

If $v(t)$ gives the velocity of of an object at time t, then the *(net) displacement*, D, of the object from $t = a$ to $t = b$ is

$$D = \int_a^b v(t)\,dt.$$

5. Suppose that the velocity of a space shuttle t seconds after takeoff is modeled by $v(t) = 0.125t^2 - 4.8t$, in m/s. This model is only valid in the first 124 seconds while the rocket boosters are assisting.

(a) What is the velocity of the shuttle after 124 seconds?

$v(124) = 1326.8\ m/s$

(b) Write down (but don't compute) a definite integral that expresses the distance traveled by the rocket in the first 124 seconds.

$$D = \int_0^{124} (0.125t^2 - 4.8t)\,dt$$

(c) Estimate the distance traveled by the rocket in the first 124 seconds using R_4 (and a calculator).

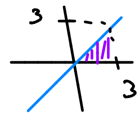
$R_4 = \left(v(31) + v(62) + v(93) + v(124)\right)\cdot 31$

$= \boxed{65588.25\ m}$

Actual distance is $\approx$ 42 km ... in 124 sec... wow!

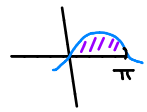
4.9, 5.3, 5.4 FTC and Antiderivatives

Q: what is $\int_0^3 x\,dx$?



$\int_0^3 x\,dx = \frac{9}{2}$

Q: what is $\int_0^\pi \sin x\,dx$?



$\int_0^\pi \sin x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \ldots$

this is a lot of work!
can we speed this up?

Fundamental Theorem of Calculus (Part 2)

WS-28

If f is continuous on $[a,b]$ and F is any antiderivative for f, i.e. $F' = f$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Wow!! ... and why?!

Ex Compute $\int_0^\pi \sin x\,dx$.

1. Need an antider. for $\sin x$.

   $\ldots (\cos x)' = -\sin x \ldots (-\cos x)' = \sin x$

   so we use $F(x) = -\cos x$

2. $$\int_0^\pi \sin x\,dx = F(\pi) - F(0) = \left[-\cos(\pi)\right] - \left[-\cos(0)\right] = \boxed{2}$$

We now pause to practice antiderivatives.

Antiderivatives

GW-28

Def F is an antiderivative for f if $F' = f$.

Ex Find an antiderivative for $f(x) = x^2$. How many can you find?

$\frac{1}{3}x^3, \quad \frac{1}{3}x^3 + 1, \quad \frac{1}{3}x^3 + \pi, \ \ldots$

$\boxed{\text{GW-28}}$ All ... but pause for indef. integral and antider. of powers.

$\int f(x)\,dx$ represents a family of functions.

$\int_a^b f(x)\,dx$ is a number, representing an area.

Properties of Indef. Integrals

- $\int c f(x)\,dx = c \int f(x)\,dx$
- $\int \left(f(x) \pm g(x)\right) dx = \int f(x)\,dx \pm \int g(x)\,dx$

Back to FTC-2

$\boxed{\text{GW-29}}$ #1, 2

Question: To use FTC-2, we need to find antiderivatives. Can this always be done?

Thinking out loud... antider. are connected to definite integrals... def. integrals are connected to area... so let's try returning to area...

$\boxed{\text{GW-29}}$ #3

Fundamental Theorem of Calculus (Part 1)

If f is continuous on $[a,b]$, then the area function

$$A(x) = \int_a^x f(t)\,dt$$

is an antiderivative for f on (a,b). That is, $A'(x) = f(x)$.

More is true...

1. If c is <u>any</u> number in $[a,b]$, then

$$\frac{d}{dx}\left[\int_c^x f(t)\,dt\right] = f(x).$$

2. More generally,

$$\frac{d}{dx}\left[\underbrace{\int_c^{\overset{u}{h(x)}} f(t)\,dt}_{A(u)=A(h(x))}\right] = f(h(x))\cdot h'(x)$$

chain rule

Ex If $g(x) = \int_5^{\overset{u}{x^3}} \sin(t)\,dt$, then find $g'(x)$.

By FTC-1, $g'(x) = \sin(u)\cdot u' = \boxed{\sin(x^3)\cdot 3x^2.}$

Ex Find an antiderivative for $f(x) = e^{x^2}$.

Let $A(x) = \int_0^x e^{t^2}\,dt$. By FTC-1, $A'(x) = e^{x^2}$, so

A is an antiderivative for f.

* This is not a very satisfying formula — we will do better in Calc 2.

OPTIONAL

pf sketch for FTC-1

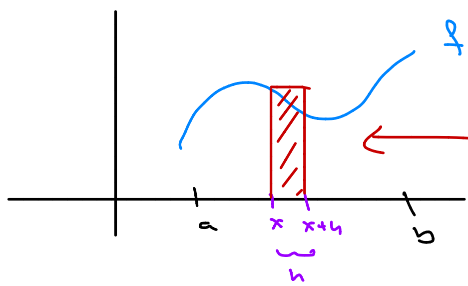
Let f be continuous on $[a,b]$. Define

$$A(x) = \int_a^x f(t)\,dt.$$

We want to show $A'(x) = f(x)$. Now,

$$A'(x) = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h}$$

Let's study the numerator.



$$A(x+h) - A(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$$

$$= \underbrace{\int_x^{x+h} f(t)\,dt}_{\text{the red area}}$$

$$\approx f(x)\cdot h$$

Thus,

$$A'(x) = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h} = \lim_{h\to 0} \frac{f(x)\cdot \cancel{h}}{\cancel{h}} = f(x).$$

$\square$

OPTIONAL

pf sketch for FTC-2

Let f be continuous on $[a,b]$. Let F be <u>any</u> antiderivative for F. We want to show

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

By FTC-2,

$$A(x) = \int_a^x f(t)\,dt$$

is another antiderivative for f. Thus,

$$F(x) = A(x) + C.$$

Now,

$$\begin{aligned} F(b) - F(a) &= (A(b) + C) - (A(a) + C) \\ &= A(b) - A(a) \\ &= \int_a^b f(t)\,dt - \cancelto{0}{\int_a^a f(t)\,dt} \\ &= \int_a^b f(x)\,dx. \quad \square \end{aligned}$$

NET Change (Interpreting definite integrals)

Recall FTC-2

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Thus,

$$\boxed{\int_a^b \begin{pmatrix}\text{rate of}\\ \text{change}\end{pmatrix} dx = \begin{pmatrix}\text{net change}\\ \text{over } [a,b]\end{pmatrix}}$$

Ex

① Suppose $s(t)$ gives position at time t.

$$\int_{t_1}^{t_2} \underbrace{s'(t)}_{\text{velocity}}\, dt = \underbrace{s(t_2) - s(t_1)}_{\text{displacement}}$$

① Suppose $P(t)$ is the size of a population at time t.

$$\int_{t_1}^{t_2} \underbrace{P'(t)}_{\text{growth rate}}\, dt = \underbrace{P(t_2) - P(t_1)}_{\text{change in population}}$$

Ex Suppose an object attached to a spring moves with velocity

$$v(t) = \sin\left(\tfrac{\pi}{2} t\right) \text{ ft/s}.$$

(a) What is the displacement of the object from $t=1$ to $t=6$?

$$s(6) - s(1) = \int_1^6 s'(t)\,dt = \int_1^6 \sin\left(\tfrac{\pi}{2}t\right) dt$$

$$= -\frac{2}{\pi}\cos\left(\tfrac{\pi}{2}t\right)\Big|_1^6 = \boxed{\frac{2}{\pi} \text{ ft}}$$

(b) What is the displacement from $t=2$ to $t=6$?

$$s(6) - s(2) = \int_2^6 \sin\left(\tfrac{\pi}{2}t\right) dt = -\frac{2}{\pi}\cos\left(\tfrac{\pi}{2}t\right)\Big|_2^6 = \boxed{0 \text{ ft}}$$

$\boxed{\text{GW-30}}$ #1

5.5 u-substitution

Computing antiderivatives is important! We need more techniques, so we can solve more problems.

<u>Ex</u> Compute $\int \cos(x^2)\cdot x\,dx$

Option 1: Trial & Error

Need to find $F(x)$ s.t. $F'(x) = \cos(x^2)\cdot x$

Try $F(x) = \sin(x^2) \xrightarrow{\text{der}} \cos(x^2)\cdot 2x$ No :(

Try $F(x) = \frac{1}{2}\sin(x^2) \xrightarrow{\text{der}} \cos(x^2)\cdot x$ Yes!

$$\int \cos(x^2)\cdot x\,dx = \boxed{\tfrac{1}{2}\sin(x^2) + C}$$

Option 2: substitution

Let $u = x^2$.

$$\int \cos(x^2)\cdot x\,dx = \int \cos(u)\cdot \underline{x\,dx}$$

need to fully substitute for x

- $u = x^2 \Rightarrow \frac{du}{dx} = 2x \Rightarrow \boxed{du = 2x\,dx}$
- thus, $x\,dx = \frac{1}{2}du$

$$\int \cos(x^2)\,x\,dx = \int \cos(u)\cdot\tfrac{1}{2}du = \tfrac{1}{2}\int \cos u\,du$$

$$= \tfrac{1}{2}\sin u + C = \boxed{\tfrac{1}{2}\sin(x^2) + C}$$

substitute back

Substitution Rule

If $u = g(x)$, then $du = g'(x)\,dx$ and

$$\int f(g(x)) \cdot g'(x)\,dx = \int f(u)\,du.$$

Choosing u

1. Often want u to be the "inside" of a function
   - this could also mean u should be the exponent or the bottom of a fraction
   - this does NOT always work
2. Might have to try multiple substitutions

Ex Compute $\int 7x^2 \sqrt{1-x^3}\,dx$

Try $u = 1-x^3 \implies \frac{du}{dx} = -3x^2 \implies du = -3x^2\,dx \implies -\frac{1}{3x^2} \cdot du = dx$

$$\int 7x^2 \sqrt{1-x^3}\,dx = \int 7x^2 \sqrt{u}\,dx$$

$$= \int 7\cancel{x^2}\sqrt{u} \cdot \frac{-1}{3\cancel{x^2}} \cdot du$$

$$= -\frac{7}{3} \int u^{1/2}\,du$$

$$= -\frac{7}{3} \cdot \frac{2}{3} \cdot u^{3/2} + C$$

sub. back

$$= \boxed{-\frac{14}{9}\left(1-x^3\right)^{3/2} + C}$$

Substitution with Definite Integrals

Ex Compute $\int_0^4 \sqrt{2x+1}\, dx$

Try $u = 2x+1 \Rightarrow du = 2\,dx \Rightarrow \frac{1}{2}\,du = dx$

$$\int_0^4 \sqrt{2x+1}\, dx = \int_{x=0}^{x=4} \sqrt{2x+1}\, dx$$

$$= \int_{x=0}^{x=4} \sqrt{u} \cdot \frac{1}{2}\, du$$

$u = 2x+1$

$x = 4 \longrightarrow u = 9$

$x = 0 \longrightarrow u = 1$

$$= \frac{1}{2} \int_{u=1}^{u=9} u^{1/2}\, du$$

$$= \frac{1}{2}\,\frac{2}{3}\, u^{3/2} \Big|_1^9$$

$$= \frac{1}{3}\, 9^{3/2} - \frac{1}{3}\, 1^{3/2} = \frac{1}{3}\cdot 27 - \frac{1}{3} = \boxed{\frac{26}{3}}$$

Ex Compute $\int_{\pi/2}^{\pi} \sin^2\theta \cos\theta \, d\theta$.

Try
$u = \sin\theta$
$du = \cos\theta\, d\theta$

$$\int_{\pi/2}^{\pi} (\sin\theta)^2 \cos\theta\, d\theta = \int_{\theta=\pi/2}^{\theta=\pi} u^2 \cdot du$$

$\theta = \pi \longrightarrow u = 0$

$\theta = \pi/2 \longrightarrow u = 1$

$$= \int_1^0 u^2\, du$$

$$= \frac{1}{3} u^3 \Big|_1^0$$

$$= 0 - \frac{1}{3}$$

$$= \boxed{-\frac{1}{3}}$$

More Practice

Ex Compute $\int \frac{2x^3}{1-x^4}\,dx$

try $u = 1-x^4$

$du = -4x^3\,dx$

$-\frac{1}{4x^3}\,du = dx$

$$\int \frac{2x^3}{1-x^4}\,dx = 2\int \frac{x^3}{u}\,dx$$

$$= 2\int \frac{\cancel{x^3}}{u}\cdot\frac{-1}{4\cancel{x^3}}\cdot du$$

$$= -\frac{1}{2}\int \frac{1}{u}\,du$$

$$= -\frac{1}{2}\ln|u| + C$$

$$= \boxed{-\frac{1}{2}\ln|1-x^4| + C}$$

WS-30 #2, 3

Ex Compute $\int \frac{x}{\sqrt{1-x^4}}\,dx$

~~1st try: $u = 1-x^4$ ... doesn't work~~

~~2nd try: $u = \sqrt{1-x^4}$ ... doesn't work~~

3rd try: $u = x^2$

$du = 2x\,dx$

$\frac{1}{2x}\,du = dx$

$$\int \frac{x}{\sqrt{1-x^4}}\,dx = \int \frac{x}{\sqrt{1-u^2}}\,dx$$

$$= \int \frac{x}{\sqrt{1-u^2}}\cdot\frac{1}{2x}\,du$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \frac{1}{2}\arcsin(u) + C$$

$$= \boxed{\frac{1}{2}\arcsin(x^2) + C}$$

Ex $\int x^2\sqrt{1-x}\,dx$

try $u=1-x$

$du=-dx$

$$\int x^2\sqrt{1-x}\,dx = \int x^2\sqrt{u}\,dx$$

$$= -\int x^2\sqrt{u}\,du$$

what to do?

... reuse the substitution!

$u=1-x \Rightarrow x=1-u$

$$= -\int (1-u)\sqrt{u}\,du$$

$$= -\int (1-u)u^{1/2}\,du$$

$$= -\int u^{1/2} - u^{3/2}\,du$$

$$= -\tfrac{2}{3}u^{3/2} - \tfrac{2}{5}u^{5/2} + C$$

$$= \boxed{-\tfrac{2}{3}(1-x)^{3/2} - \tfrac{2}{5}(1-x)^{5/2} + C}$$

Thm $\int \tan\theta\,d\theta = -\ln|\cos\theta| + C = \ln|\sec\theta| + C$

pf

$u=\cos\theta$

$du=-\sin\theta\,d\theta$

$$\int \tan\theta\,d\theta = \int \frac{\sin\theta}{\cos\theta}\,d\theta$$

$$= -\int \frac{1}{u}\,du$$

$$= -\ln|u| + C$$

$$= \boxed{-\ln|\cos\theta| + C}$$

$$= \ln|(\cos\theta)^{-1}| + C$$

$$= \boxed{\ln|\sec\theta| + C}$$

$\square$