EXAM 2 - REVIEW QUESTIONS

MODERN ALGEBRA

Questions (answers are on page 3)

Examples. For each of the following, either provide a specific example which satisfies the given description, or if no such example exists, briefly explain why not.

(1) (JH) Integral domain that is not a field.
(2) (JB) A polynomial of degree 5 that is irreducible over $\mathbb{Q}$ by EIC with the coefficient of x^2 equal to 1
(3) (JB_2) An ideal of $\mathbb{Z}[x]$
(4) (JT) A polynomial of degree at least 4 that fails EIC and is reducible
(5) (TB) Let K be an extension field of $\mathbb{Q}$. Give an example of K where K is not a vector space over $\mathbb{Q}$.
(6) (PS) Nonzero polynomial $p(x), a(x), b(x) \in \mathbb{Q}[x]$ such that $p(x)$ divides $a(x)b(x)$ but $p(x)$ does not divide $a(x)$ or $b(x)$
(7) (JR) A prime that makes $p(x) = x^4 + 30x^3 + 60x^2 + 30x + 900$ irreducible by EIC
(8) (ST) A polynomial of degree 20 that is irreducible over $\mathbb{Q}$
(9) (NL) Give a set that is a noncommutative ring with unity
(10) (OA) Provide three vector spaces L, K, F such that $L \subseteq K \subseteq F$ and the dimension of F over L is twice the dimension of K over L
(11) (RM) Provide an example of a subring that is not an ideal.
(12) (MM) Give an example of a reducible polynomial of degree 5.
(13) (PC) A number $\sqrt[n]{m} \in \mathbb{Q}(\sqrt[p]{q})$ where $n > p$, m, q are prime, and $n, m, p, q \in \mathbb{Z}$.
(14) (ZA) A non trivial ideal of a noncommutative ring
(15) (EH) Give an example of a principle ideal domain
(16) (ED) Two polynomials contained in an ideal I of $\mathbb{Q}[x]$ that guarantee that $I = \mathbb{Q}[x]$.
(17) (MJ) Give an example of a ring that is not an integral domain.
(18) (JM) A polynomial of the form $ax^3 + bx^2 + cx + d$ such that d divides a that is irreducible by EIC.
(19) (AJ) A minimal polynomial for $\sqrt[3]{2}$ over $\mathbb{Q}$ other than $m(x) = x^3 - 2$.
(20) (LK) A ring, R, with $a, x, y \in R$ and $a \neq 0$ such that $ax = ay$ or $xa = ya$ and $x \neq y$.
(21) (CH) Find a minimal polynomial such that $[\zeta_5 : \mathbb{Q}] = 5$.
(22) (JD) A set R that is a ring with unity, but has no further properties.
(23) (AB) Give an example of a commutative ring that is not a ring with unity.
(24) (KH) Give an example of an ideal that is not principle.

True or False.

(25) (JH) $\sqrt[3]{2} \in \mathbb{Q}(\sqrt{2})$
(26) (JB) For all n, the minimum polynomial of ζ_n has degree $n-1$
(27) (JB_2) Let $I \subseteq \mathbb{Q}[x]$ be an ideal. For $p(x), q(x) \in I$ and $\gcd(p(x), q(x)) = 1$, $I = \mathbb{Q}[x]$
(28) (JT) All ideals are principal ideals
(29) (PS) The even integers form an ideal of $\mathbb{Z}$
(30) (JR) $\mathbb{Z}_4[x]$ is an integral domain
(31) (ST) If $4 = [r : \mathbb{Q}]$ then $\mathbb{Q}(r) = \{a_0 + a_1 r + a_2 r^2 + a_3 r^3 + a_4 r^4 | a_i \in \mathbb{Q}\}$
(32) (NL) $x^6 + 4x + 4$ is irreducible in $\mathbb{Q}[x]$ by EIC with $p = 4$
(33) (OA) Let $I = \{g(x) \in \mathbb{Q}[x] | g(\sqrt[3]{5}) = 0\}$. Then I is an ideal in $\mathbb{Q}[x]$
(34) (RZ) $F[x]$ is an integral domain if F is a ring
(35) (RM) $\mathbb{Z}[x]$ is a principal ideal domain.

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(36) (MM) A vector space is nontrivial if it has at least one element.
(37) (PC) The empty set is an ideal of $\mathbb{Z}$.
(38) (ZA) Every Ring has an ideal.
(39) (EH) $p(x) = x^7 + 3x^5 + 12x^4 + 6x^3 + 3x^2 + 9$ is reducible in $\mathbb{Q}[x]$.
(40) (ED) A commutative ring does not need to have a unit element and a ring with a unit element does not need to be commutative.
(41) (MJ) $p(x) = x^2 + 1$ is irreducible over $\mathbb{Z}_2$.
(42) (JM) Let F be a subfield of E and let $r \in E$. Then any polynomial $p(x) \in F[x]$ such that $p(r) = 0$ can be used to show r is algebraic over F.
(43) (SL) The polynomial $p(x) = x^7 + 2 + 2$ is irreducible over $\mathbb{Q}$.
(44) (AJ) The polynomial $p(x) = x^5 - 3x^4 + 3x^2 - 3$ is irreducible over $\mathbb{Q}(\sqrt{2})$ by EIC with $p = 3$.
(45) (LK) Let F be a field and D be an integral domain such that $F \subseteq D$. Then D is a vector space over F.
(46) (CH) Eisenstein's Irreducibility Criterion applies to polynomials in $\mathbb{Q}[x]$.
(47) (JD) $\mathbb{R}$ is both a subring and an ideal of $\mathbb{C}$

Answers

1. (1) $\mathbb{Q}[x]$
2. (2) Impossible, because no prime divides 1
3. (3) $I = (2\mathbb{Z})[x]$
4. (4) $x^4 - 1 = (x^2+1)(x^2-1)$ and no prime divides 1
5. (5) Not possible. If E is an extension of a field F then E is always a vector space over F
6. (6) $p(x) = x^2 - 4$, $a(x) = x - 2$, $b(x) = x + 2$
7. (7) Not possible. If p divides 30 then p^2 divides 900
8. (8) $p(x) = x^{20} - 2$
9. (9) $\mathbb{H}$
10. (10) $\mathbb{Q} = L$, $\mathbb{Q}(\sqrt{2}) = K$, $\mathbb{Q}(\sqrt{2}, i) = F$
11. (11) $\mathbb{Z}[x] \subset \mathbb{Q}[x]$
12. (12) $x^5 - 1 = (x-1)(x^4 + x^3 + x^2 + x + 1)$
13. (13) No such example exists by Prop 12.5 or Prop 12.10
14. (14) $M_{2x2}(2\mathbb{Z}) \subset M_{2x2}(\mathbb{Z})$
15. (15) $\mathbb{Z}$
16. (16) $p(x) = x^4 + x^3 + x^2 + x + 1$ and $q(x) = x^6 + 4x^4 - 6x^3 + 2$
17. (17) $2\mathbb{Z}$, because it doesn't have unity 1
18. (18) Impossible, because if some prime p divides d (and if d divides a), then p will divide a and thus fail EIC.
19. (19) Impossible. The minimal polynomial for any r over F is unique.
20. (20) $\mathbb{Z}_6$
21. (21) Impossible, $x^5 - 1$ can be reduced to $x^4 + x^3 + x^2 + x + 1$ which has a degree of 4.
22. (22) $R = M_{2x2}(\mathbb{R})$
23. (23) $2\mathbb{Z}$
24. (24) $I = \{2p(x) + xq(x) | p(x), q(x) \in \mathbb{Z}[x]\} \subset \mathbb{Z}[x]$
25. (25) F, compare $[\sqrt[3]{2} : \mathbb{Q}]$ and $[\sqrt{2} : \mathbb{Q}]$ and use 12.5 or 12.10
26. (26) F, $\zeta_4 = i$ and the min. poly. is $x^2 + 1$
27. (27) T, because $\gcd(p(x), q(x)) \in I$ and if $1 \in I$ then I is the whole ring.
28. (28) F, consider the ring $\mathbb{Z}[x]$ and the ideal $\{2p(x) + xq(x) | p(x), q(x) \in \mathbb{Z}[x]\}$
29. (29) T
30. (30) F, $[2]_4 * [2]_4 = 0$
31. (31) T, BUT note that $\{1, r, \ldots, r^4\}$ is NOT a basis for $\mathbb{Q}(r)$ over $\mathbb{Q}$
32. (32) F, because 4 is not a prime
33. (33) T, just use the Ideal Criteria Prop.
34. (34) F, $\mathbb{Z}_4[x]$ is not an integral domain because $[2]_4 * [2]_4 = 0$
35. (35) F, $I = \{2p(x) + xq(x) | p(x), q(x) \in \mathbb{Z}[x]\}$ is not principle.
36. (36) F, a vector space is nontrivial if it has at least two elements.
37. (37) F, the empty set is not an ideal of $\mathbb{Z}$ by Prop 9.2.
38. (38) T, the set containing just zero is always an ideal
39. (39) ?, EIC doesn't apply, so don't know anything.
40. (40) T, see Chapter 7 Page 98
41. (41) F, $p(x) = x^2 + 1 = (x+1)^2$ in $\mathbb{Z}_2$.
42. (42) F, if $p(x) = 0$ then $p(r) = 0$, but you must use a nonzero polynomial to show r is algebraic.
43. (43) T, apply EIC with $p = 2$.
44. (44) F, EIC cannot be directly applied to show the polynomial is irreducible over $\mathbb{Q}(\sqrt{2})$, as EIC only guarantees irreducibility over $\mathbb{Q}$.
45. (45) T, similar explanation to #5 but for integral domain
46. (46) F, EIC can only be applied to polynomials in $\mathbb{Z}[x]$.
47. (47) F, $\mathbb{R}$ is not an ideal of $\mathbb{C}$