EXAM 3 - REVIEW QUESTIONS

MODERN ALGEBRA

Questions (answers are on page 2)

Examples. For each of the following, either provide a specific example which satisfies the given description, or if no such example exists, briefly explain why not.

(1) (CK) A subgroup of S_5 containing a transposition and a 5-cycle such that $S \neq S_5$
(2) (JH) An nonprime integer n such that S_n is generated by any subset containing an n-cycle and a transposition.
(3) (JB) A group that contains only two elements.
(4) (WS) An example of a nonabelian group with exactly 8 elements
(5) (JT) A homomorphism of groups that is one-to-one but not onto
(6) (PS) An element of S_8 of order 8 that is not an 8-cycle.
(7) (HD) A homomorphism that is onto but not one-to-one
(8) (AW) A group that is not cyclic
(9) (NL) A cyclic group that is nonabelian.
(10) (MG) Two disjoint cycles in S_6 that do not commute
(11) (EH) An element g in S_8 where $\mid g \mid= 5$
(12) An abelian group.
(13) (AB) An element in S_7 that has order greater than 7.
(14) (PC) Give an example where $[K : \mathbb{Q}]$ is finite and $|GalK/\mathbb{Q}| \neq [K : \mathbb{Q}]$.

True or False.

(15) (JH) Let f' denote the usual derivative of a function. If $d : \mathbb{R}[x] \to \mathbb{R}[x] : f \mapsto f'$, then d is onto.
(16) (JB) If a group G has infinite order, then the only element of G that does not have infinite order is the identity.
(17) (WS) If a group contains only 2 elements then it is abelian.
(18) (JT) The number of elements in $\text{Gal}(\mathbb{Q}(\sqrt[5]{7}, \zeta_5)/\mathbb{Q})$ is 25.
(19) (HD) There are at least 2 subgroups of every group.
(20) (OA) All infinite groups are noncyclic.
(21) (RZ) If I is an ideal of a ring R, then $(I, +)$ is always an abelian group.
(22) (NL) The group $(\mathbb{H}^\times, \cdot)$ is nonabelian.
(23) (MG) S_{21} is generated by any 21-cycle together with (23)
(24) (MG) S_{21} is generated by some 21-cycle together with (23)
(25) (RM) If A and B are subgroups, then $A \cap B$ is a subgroup of G.
(26) If F is a subfield of K and $r \in K$ is algebraic over F, then $|GalF(r)/F| = [F(r) : F]$.
(27) (2135) and (4627) commute.
(28) (AB) $h : R \to C$, where, $h(r) = r + 0i$ is an automorphism.
(29) (PC) There exists an isomorphism $f : \mathbb{Q}(\sqrt{3}) \to \mathbb{Q}(\sqrt{5})$.

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Answers

(1) Not possible, see Ch. 19
(2) Not possible, see Ch. 19, i.e. the property holds iff n is prime.
(3) S_2 or $(\{\pm 1\}, \cdot)$ or $(\mathbb{Z}_2, +)$
(4) $Q = \{\pm 1, \pm i, \pm j, \pm k\}$
(5) $f : (\mathbb{R}, +) \to (\mathbb{C}, +) : x \mapsto x$
(6) Not possible, think about the lcm of numbers less than 8
(7) $f : (\mathbb{Z}, +) \to (\mathbb{Z}_7, +) : x \mapsto [x]_7$ or $g : (\mathbb{C}, +) \to (\mathbb{R}, +) : x + yi \mapsto x$
(8) $(\mathbb{Q}, +)$ or S_3
(9) Not possible, see Ch. 21
(10) Not possible
(11) (12345). But as a follow-up, S_8 has no element of order 25 by proposition 21.16
(12) $(\mathbb{Z}, +)$
(13) (12)(34567) has an order of 10.
(14) $K = \mathbb{Q}(\sqrt[3]{2})$
(15) T
(16) F, consider -1 in $(\mathbb{R}^{\times}, \cdot)$
(17) T
(18) F, it's 20. Notice that $\mathbb{Q}(\sqrt[5]{7}, \zeta_5) = \mathbb{Q}^{x^5-7}$ so Prop. 14.8 applies.
(19) F, not for the "trivial" group with one element
(20) F, consider $(\mathbb{Z}, +)$; it is generated by 1.
(21) T
(22) T
(23) F, 21 is not prime.
(24) T
(25) T
(26) Not always true. See the answer to (14). However, it is true that $|GalF^{p(x)}/F| \leq [F(r) : F]$
(27) False
(28) False, it's not onto.
(29) False