03 – Vector Spaces

Definition: Vector Space

A real **vector space** (or a **vector space** over $\mathbb{R}$) is a set V with two operations $\oplus$ and $\odot$ and a distinguished element $\mathbf{0}$ such that the following axioms hold for <u>all</u> $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and <u>all</u> $c, d \in \mathbb{R}$:

(a) [$\oplus$ CLOSURE] $\mathbf{u} \oplus \mathbf{v} \in V$

- **(1)** [$\oplus$ COMMUTATIVITY] $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}$
- **(2)** [$\oplus$ ASSOCIATIVITY] $\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}$.
- **(3)** [$\oplus$ IDENTITY] $\mathbf{u} \oplus \mathbf{0} = \mathbf{u} = \mathbf{0} \oplus \mathbf{u}$
- **(4)** [$\oplus$ INVERSES] there is a $\mathbf{u}' \in V$ such that $\mathbf{u} \oplus \mathbf{u}' = \mathbf{0} = \mathbf{u}' \oplus \mathbf{u}$
  - we typically write $-\mathbf{u}$ for the element $\mathbf{u}'$

(b) [$\odot$ **closure**] $c \odot \mathbf{u} \in V$

- **(5)** [DISTRIBUTIVITY OF $\odot$ OVER $\oplus$] $c \odot (\mathbf{u} \oplus \mathbf{v}) = (c \odot \mathbf{u}) \oplus (c \odot \mathbf{v})$
- **(6)** [$\odot$ COMPATIBILITY WITH $+$] $(c + d) \odot \mathbf{u} = (c \odot \mathbf{u}) \oplus (d \odot \mathbf{u})$
- **(7)** [$\odot$ COMPATIBILITY WITH $\cdot$] $(c \cdot d) \odot \mathbf{u} = c \odot (d \odot \mathbf{u})$
- **(8)** [$\odot$ COMPATIBILITY WITH 1] $1 \odot \mathbf{u} = \mathbf{u}$

The elements of a vector space will be called **vectors**; the elements of $\mathbb{R}$ will be called **scalars**. The operation $\oplus$ is called **vector addition** and $\odot$ is called **scalar multiplication**.

1. Let $V = \mathbb{R}^n$. Define $\oplus$ to be usual addition of vectors and $\odot$ to be usual scalar multiplication (as defined in Chapter 1). Explain why V is a vector space with respect to these operations.

2. Let $V = \mathbb{R}^2$. Define $\oplus$ to be usual addition of vectors, and define $\odot$ via $c \odot \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} x \\ cy \end{smallmatrix}\right]$. Prove that V is **not** a vector space with respect to these operations.

Example

Each of the following is an example of a vector space.

1. Fix $m, n \geq 1$. Let M_{mn} be the set of all $m \times n$ matrices with entries from $\mathbb{R}$. Define $\oplus$ to be usual addition of matrices and $\odot$ to be usual scalar multiplication.

2. Fix $n \geq 0$. Let P_n be the set of all polynomials—with coefficients in $\mathbb{R}$—of *degree at most* n, together with the zero polynomial. Define $\oplus$ to be usual addition of polynomials and $\odot$ to be usual multiplication of a real number by a polynomial.

3. Let P be the set of all polynomials—with coefficients in $\mathbb{R}$—of any degree, together with the zero polynomial. Define $\oplus$ to be usual addition of polynomials and $\odot$ to be usual multiplication of a real number by a polynomial.

3. Let V be the set of *positive* real numbers. For $\mathbf{u}, \mathbf{v} \in V$ and $c \in \mathbb{R}$, define $\mathbf{u} \oplus \mathbf{v} = \mathbf{u} \cdot \mathbf{v}$ (so $\oplus$ is multiplication) and $c \odot \mathbf{v} = \mathbf{v}^c$. Prove that V is a vector space with respect to these operations.

Theorem: General Properties of Vector Spaces

Let V be a vector space. Then the following hold for all $\mathbf{u} \in V$ and all $c \in \mathbb{R}$:

(a) $0 \odot \mathbf{u} = \mathbf{0}$

(b) $c \odot \mathbf{0} = \mathbf{0}$

(c) If $c \odot \mathbf{u} = \mathbf{0}$, then $c = 0$ or $\mathbf{u} = \mathbf{0}$

(d) $(-1) \odot \mathbf{u} = -\mathbf{u}$

4. Prove Property **(d)** of the theorem using the definition of a vector space together with Property **(a)**.