3. Group actions

"Groups, as men, will be known by their actions."
- Guillermo Moreno

3.1. **The definition.**

Definition 3.1. An ***action*** of a group G on a set X is a function from $\alpha : G \times X \to X$ such that the following hold for all $g, h \in G$ and all $x \in X$; we write $g \cdot x$ in place of $\alpha(g, x)$.

(1) $g \cdot (h \cdot x) = (gh) \cdot x$, and
(2) $1 \cdot x = x$.

Problem 3.2. Suppose that G acts on X. Fix $g \in G$, and consider the function $\sigma_g : X \to X$ defined by $\sigma_g(x) = g \cdot x$.

(1) Show that σ_g is a bijection, i.e. show that $\sigma_g \in \mathrm{Sym}(X)$. *[Hint: make use of the fact that g has an inverse.]*
(2) Show that the function $\sigma : G \to \mathrm{Sym}(X) : g \mapsto \sigma_g$ is a homomorphism. *[Hint: in order to show that $\sigma_{gh} = \sigma_g \circ \sigma_h$, show that $\sigma_{gh}(x) = (\sigma_g \circ \sigma_h)(x)$ for all $x \in X$.]*

Remark 3.3. In the previous problem, the function σ is called the ***associated permutation representation*** of G since it gives a way to view each element of G as a permutation of X in a way "compatible" with the operations of G.

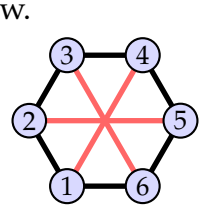
Definition 3.4. Assume that G acts on X.

(1) The action is ***transitive*** if for every $x, y \in X$ there is a $g \in G$ such that $g \cdot x = y$.
(2) For $g \in G$ and $x \in X$, we say that g ***fixes*** x if $g \cdot x = x$.
(3) For $x \in X$, the ***stabilizer of*** x, denoted G_x, is set of all $g \in G$ that fix x.
(4) For $S \subseteq X$, the ***stabilizer of*** S, denoted G_S, is set of all $g \in G$ that fix every $x \in S$.
(5) The ***kernel*** of the action is the subset of G that fixes every $x \in X$, i.e. G_X.
(6) The action is said to be ***faithful*** if the kernel is trivial.

Remark 3.5. Observe that the kernel of an action corresponds with the kernel of the associated permutation representation, so an action is faithful if and only if the associated permutation representation is injective.

Theorem⋆ 3.6. *An action of a group G on X is transitive if there exists some $x \in X$ such that for all $y \in X$ there is a $g \in G$ for which $g \cdot x = y$.*

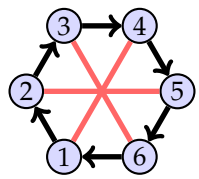
Problem 3.7. Recall that D_6 is the automorphism group of the square $\mathcal{D}_6$. Let X be the set of (three) diagonal edges shown below.



Note that for every diagonal edge $\{u, v\}$ and every $\varphi \in D_6$, the edge $\{\varphi(u), \varphi(v)\}$ is again a diagonal edge. Thus, we have an action of D_6 on X given by $\varphi \cdot \{u, v\} = \{\varphi(u), \varphi(v)\}$.

(1) Is the action transitive?
(2) Let E be the edge $E := \{1, 4\}$. Determine the stabilizer G_E.
(3) Find a relationship between $|G|$, $|X|$, and $|G_E|$.
(4) Determine the kernel of the action. Is the action faithful?
(5) What is the image of the associated permutation representation?

Problem 3.8. Repeat the previous problem for C_6. Let X be the set of (three) diagonal edges shown below.



As before, we have an action of C_6 on X given by $\varphi \cdot \{u, v\} = \{\varphi(u), \varphi(v)\}$.
(1) Is the action transitive?
(2) Let E be the edge $E := \{1, 4\}$. Determine the stabilizer G_E.
(3) Find a relationship between $|G|$, $|X|$, and $|G_E|$.
(4) Determine the kernel of the action. Is the action faithful?
(5) What is the image of the associated permutation representation?

3.2. **Action by left multiplication.**

Problem 3.9 (Action by left multiplication)**.** Let G be a group.
(1) Show that $g \cdot h = gh$ defines an action of G on G; the associate representation is called called the ***left regular representation***. Is the action transitive? For $h \in G$, determine the stabilizer of h. Is this action faithful?
(2) Let H be a subgroup of G. Show that $g \cdot aH = (ga)H$ for all $g, a \in G$ defines an action of G on the coset space G/H. Is the action transitive? Show that the stabilizer of aH is aHa^{-1}. Give an example of a group G and a proper nontrivial subgroup H for which this action is **not** faithful.

Theorem⋆ 3.10. *If G acts on X and $S \subseteq X$, then G_S is a subgroup of G, and the kernel of the action is a normal subgroup.*

Definition 3.11. A group is ***simple*** if it has *no* proper nontrivial normal subgroups.

Remark 3.12. Whenever a group G has a proper nontrivial normal subgroup N, we can break G into two "simpler" pieces: N and G/N. The simple groups are the groups that can not be broken down this way and can be thought of as the "basic" building blocks.

Theorem⋆ 3.13. *If G is an infinite group with a proper subgroup H of finite index, then G is **not** simple.* [Hint: consider the kernel of the action of G on G/H by left multiplication. The First Isomorphism Theorem, i.e. Theorem 2.84 and Remark 2.85, may also be helpful.]

Theorem⋆ 3.14. *If G is an finite group with a proper subgroup H for which $|G|$ has a prime divisor larger than the index of H, then G is **not** simple.* [Hint: same hint as the previous problem.]

3.3. **Action by conjugation.**

Notation 3.15. Let G be a group. For $g \in G$, the function $\gamma_g : G \to G$ defined by $\gamma_g(h) = ghg^{-1}$ is called ***conjugation by*** g.

Theorem★ 3.16. *If G is a group and $g \in G$, then γ_g is an automorphism of G. In particular,*

(1) if $h \in G$, then $|h| = |ghg^{-1}|$, and
(2) if H is a subgroup of G, then gHg^{-1} is a subgroup of G with $H \cong gHg^{-1}$.

Theorem★ 3.17. *Let $\sigma, \tau \in S_n$. If the disjoint cycle decomposition of σ is*

$$\begin{pmatrix} a_1 & a_2 & \dots & a_{k_1} \end{pmatrix} \begin{pmatrix} b_1 & b_2 & \dots & b_{k_2} \end{pmatrix} \cdots,$$

then the disjoint cycle decomposition of $\tau\sigma\tau^{-1}$ is

$$\begin{pmatrix} \tau(a_1) & \tau(a_2) & \dots & \tau(a_{k_1}) \end{pmatrix} \begin{pmatrix} \tau(b_1) & \tau(b_2) & \dots & \tau(b_{k_2}) \end{pmatrix} \cdots.$$

Problem 3.18 (Action by conjugation)**.** Let G be a group.

(1) Show that $g \cdot h = ghg^{-1}$ defines an action of G on G. Show that the action is not transitive unless $|G| = 1$. For $h \in G$, show that the stabilizer of h is equal to $C_G(h)$. What is the kernel of the action?
(2) Show that $g \cdot H = gHg^{-1}$ defines an action of G on the set of all subgroups of G. The stabilizer (with respect to this action) of a subgroup H is called the ***normalizer of*** H ***in*** G and is denoted $N_G(H)$.

Theorem★ 3.19. *If H is a subgroup of a group G, then H is a normal subgroup of $N_G(H)$.*

Definition 3.19.1. If A and B are subsets of a group G, we define $AB := \{ab | a \in A, b \in B\}$.

Theorem★ 3.19.2. *If H is a subgroup of a group G and K is a subgroup of $N_G(H)$, then KH is a subgroup of $N_G(H)$.*

Definition 3.20. Let G be a group acting on a set X. If $x \in X$, then the subset of X given by $Gx := \{g \cdot x | g \in G\}$ is called the ***orbit of*** x ***(under*** G***)***.

Theorem★ 3.21. *If G is a group acting on a set X, then the set of orbits forms a partition of X.*

Definition 3.22. When G acts on itself (or on its subgroups) by conjugation, the orbits are called ***conjugacy classes*** (or ***conjugacy classes of subgroups***) and two elements in the same conjugacy class are said to be ***conjugate***.

Problem 3.23. Determine the conjugacy classes of S_3. Determine the conjugacy classes of subgroups of S_3.

Problem 3.24. Determine the conjugacy classes of D_6.

Theorem★ 3.25. *Two elements of S_n are conjugate if and only if they have the same cycle type.*

Theorem★ 3.26. *If $n \geq 3$, then $Z(S_n) = \{1\}$.*

3.4. **The Orbit-stabilizer Theorem.**

Remark 3.27. Suppose that G acts on X. Observe that, for any orbit $\mathcal{O}$, the action of G on X restricts to an action of G on $\mathcal{O}$, and this latter action is now transitive. In this way, many questions about group actions can be reduced to questions about transitive group actions.

Theorem⋆ 3.28 (Orbit-stabilizer Theorem)**.** *Let G be a group acting on a set X. Then for every $x \in X$, $|Gx| = |G : G_x|$.* [Hint: construct a bijection from G/G_x to Gx.]

Notation 3.29. For a group G acting on a set X, we define $\mathrm{Fix}(G)$ to be the set of all $x \in X$ such that x fixed by *every* element of G, i.e. $\mathrm{Fix}(G)$ is the set of fixed points of G. The elements of $\mathrm{Fix}(G)$ represent the orbits of G of size 1.

Theorem⋆ 3.30. *Let G be a finite group acting on a finite set X. Let $\mathcal{O}_1, \ldots, \mathcal{O}_n$ be the orbits of G not contained in* $\mathrm{Fix}(G)$*, if any, and let $x_1, \ldots x_n \in X$ be such that $x_i \in \mathcal{O}_i$. Then*

$$|X| = |\mathrm{Fix}(G)| + \sum_{i=1}^{n} |G : G_{x_i}|.$$

[Hint: recall that the orbits of G partition X.]

Theorem⋆ 3.31. *Let P be a group of order p^k for some prime p. If P acts on a finite set X, then $|\mathrm{Fix}(P)| \equiv |X|$ modulo p.*

Theorem⋆ 3.32. *If P is a group of order p^k for some prime p, then $Z(P)$ is nontrivial.* [Hint: let P act on itself by conjugation.]

Theorem⋆ 3.33. *If P is group of order p^2 for some prime p, then P is abelian.*

Problem 3.34 (The Class Equation)**.** Let G be a finite group. Let $C_1, \ldots, C_n$ be the conjugacy classes of G not contained in $Z(G)$, if any, and let $x_1, \ldots x_n \in G$ be such that $x_i \in C_i$. Explain how Theorem 3.30 can be used to quickly deduce that

$$|G| = |Z(G)| + \sum_{i=1}^{n} |G : C_G(x_i)|.$$

Theorem⋆ 3.35. *Let G be a finite group. If p is a prime dividing $|G|$, then p divides $|C_G(g)|$ for some nontrivial $g \in G$.* [Hint: class equation.]

Theorem⋆ 3.36 (Cauchy's Theorem)**.** *Let G be a finite group. If p is a prime dividing $|G|$, then G has an element of order p.* [Hint: consider a minimal counterexample, and first show that it must have a nontrivial center.]

Problem 3.37. We should always be asking if we can generalize things. Make at least two conjectures related to generalizing (or not being able to generalize) Cauchy's Theorem.

Theorem⋆ 3.38. *Let p be a prime. If G is a finite group, then G is a p-group (see Definition 2.55) if an only if $|G| = p^k$ for some $k \in \mathbb{N}$.*

3.5. **Sylow's Theorems.**

Definition 3.39. Let p be a prime. A subgroup P of G is called a ***Sylow p-subgroup*** if P is a p-group and is not properly contained in another p-subgroup of G, i.e. P is a maximal p-subgroup of G. Let $\mathrm{Syl}_p(G)$ be the set of Sylow p-subgroups of G.

Remark 3.40. If G is a finite group of order $p^k m$ with p prime and p not dividing m, then a Sylow p-subgroup of G has order at most p^k, by Theorem 3.38.

Problem 3.41. Find a Sylow 5-subgroup of S_5.

Problem 3.42. Find a Sylow 2-subgroup of S_4. *[Hint: the maximum possible cardinality is* 8*. Do you know of a group with* 8 *elements that acts on a set of size* 4*?]*

Theorem⋆ 3.43. *Let p be a prime. If $P \in \mathrm{Syl}_p(G)$, then $gPg^{-1} \in \mathrm{Syl}_p(G)$ for all $g \in G$, so G acts on $\mathrm{Syl}_p(G)$ by conjugation.*

Theorem⋆ 3.44. *Let p be a prime. If P is a p-subgroup of a group G and Q is a p-subgroup of $N_G(P)$, then QP is a p-subgroup of $N_G(P)$.* [Hint: first show that QP/P is a p-group.]

Theorem⋆ 3.45. *Let p be a prime, and let P be a Sylow p-subgroup of a group G. If P is normal in G, then P is the only Sylow p-subgroup of G. Thus, even when P is not normal in G, P is always the unique Sylow p-subgroup of $N_G(P)$.*

Theorem⋆ 3.46 (Sylow's Thereom - part 1)**.** *If G is a finite group and p is a prime dividing $|G|$, then G acts transitively by conjugation on $\mathrm{Syl}_p(G)$, and further, $|\mathrm{Syl}_p(G)| \equiv 1$ modulo p.*
[Hint: let $\mathcal{O}$ be some orbit of G on $\mathrm{Syl}_p(G)$. The goal is to show that $\mathcal{O} = \mathrm{Syl}_p(G)$ and that $|\mathcal{O}| \equiv 1 \bmod p$. Let $P \in \mathrm{Syl}_p(G)$ be arbitrary, and consider how P acts on $\mathcal{O}$ (by conjugation).

(1) Show that if $P \in \mathcal{O}$, then P fixes, i.e. normalizes, only P. Conclude that $|\mathcal{O}| \equiv 1$ modulo p.
(2) Show that if $P \notin \mathcal{O}$, then P fixes nothing in $\mathcal{O}$. Conclude from this that $|\mathcal{O}| \equiv 0$ modulo p.

The previous theorem and Theorem 3.31 are very relevant.]

Theorem⋆ 3.47 (Sylow's Thereom - part 2)**.** *If G is a finite group and $|G| = mp^k$ with p prime and p not dividing m, then $|P| = p^k$ for every $P \in \mathrm{Syl}_p(G)$.*
[Hint: use part 1 of Sylow's Theorem and the Orbit-Stabilizer Theorem to show $|N_G(P)| = m'p^k$ for some m'. Now, towards a contradiction, assume that $|P| = p^\ell$ with $\ell < k$, and consider the quotient group $N_G(P)/P$. Show that $N_G(P)/P$ must have an element of order p and use this find a contradiction.]

Remark 3.48. Since all Sylow p-subgroups of a finite group are conjugate, a finite group has a normal Sylow p-subgroup if and only if it has a unique one. Thus, the condition "$|\mathrm{Syl}_p(G)| \equiv 1$ modulo p" can be helpful in determining if a group has a normal Sylow

subgroup or not. And one should always remember that $|\mathrm{Syl}_p(G)| = |G : N_G(P)|$ by the Orbit-Stabilizer Theorem, so in particular, $|\mathrm{Syl}_p(G)|$ is always coprime to p.

Theorem 3.49. *If G is a group of order mq where q is prime and $m < q$, then G has a normal Sylow q-subgroup.*

Theorem 3.50. *If G is a group of order pqr where p, q, and r are prime with $p < q < r$, then some Sylow subgroup of G is normal.* [Hint: the following counting technique often works well when the largest prime divisors of $|G|$ only occur to the first power (make sure you see when you use this). The rough idea is that if no Sylow subgroup of G is normal, then G will have too many Sylow subgroups and, in turn, too many elements. Assume the theorem is false. First count the number of Sylow r-subgroups, and use this to count the number of elements of G of order r. Now estimate (it will be hard to precisely count) the number of Sylow q-subgroups, and use this to estimate the number of elements of G of order q. Finally, compare the sum of these with the order of G.]

Problem 3.51 (Just for fun)**.** Prove that every group of order 36 has a normal Sylow subgroup. *[Hint: I think this problem is hard.]*

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